M953, Homework 5, due Friday, March 15, 2013

Instructions: Do any three problems.

(1) Give and justify an example of a rational map $\phi : \mathbf{A}^1(\mathbf{R}) - - \to \mathbf{A}^1(\mathbf{R})$ which is not a morphism but for which $\operatorname{ind}(\phi) = \varnothing$.

(2) Let k be a field. Let $k[x]$ be a polynomial ring in an indeterminate x. Show that every nonconstant element $f \in k(x)$ is transcendental over k (i.e., show that there is no nonzero monic element $g \in k[t]$ having f as a root).

(3) Let $V \subseteq \mathbf{A}^m(k)$ and $W \subseteq \mathbf{A}^n(k)$ be irreducible algebraic varieties over an algebraically closed field k. Let $\phi = (\phi_1, \ldots, \phi_n) : V - - \to W$ be a rational map defined by elements $a_i/b_i = \phi_i \in k(V)$, for elements $a_i, b_i \in k[V]$ with $b_i \neq 0$. Let U be a nonempty open subset of V on which ϕ is defined (i.e., such that none of the elements b_i vanishes at any point of U). If $\phi(U)$ is dense in the Zariski topology on W, we say ϕ is *dominant*.

(a) If ϕ is dominant (i.e., if $\phi(U)$ is dense in W for some nonempty open subset $U \subseteq V$ disjoint from $V(b_1 \cdots b_n)$), show that $\phi^* : k[W] \to k(V)$ is injective, and hence induces a homomorphism $k(W) \to k(V)$.

(b) Let ϕ and U be as in (a). If $\phi(U)$ is dense, show that $\phi(U')$ is dense for every nonempty open subset $U' \subseteq V \setminus V(b_1 \cdots b_n)$.

(4) An irreducible algebraic variety $V \subseteq \mathbf{A}^n(k)$ such that $k(V)$ is a purely transcendental extension of k is said to be *rational*. If W is an irreducible algebraic variety for which there is a dominant rational map $V - - \to W$ with V rational, we say W is *unirational*.

(a) If V is rational, show that V is birationally equivalent to $\mathbf{A}^d$, where d is the transcendence degree of $k(V)$ over k.

(b) If W is unirational such that $k(W)$ has transcendence degree d over k, show that there is a dominant rational map $\mathbf{A}^d \to W$. [Hint: Show there is a dominant rational map $\mathbf{A}^n - - \to W$ for some $n \geq d$. Then show there is a linear subspace $L_d \subseteq \mathbf{A}^n$ of dimension d such that the composition $L_d \subseteq \mathbf{A}^n - - \to W$ is defined and dominant.]

(c) If W is unirational such that $k(W)$ has transcendence degree 1 over k, show that W is rational. (You may look up and apply Lüroth's Theorem without proof, but state it if you do.)

(5) Let $f, g \in k(V)$ for an irreducible algebraic variety $V \subseteq \mathbf{A}^m(k)$. Assume $f = \frac{a}{b}$ and $g = \frac{c}{d}$ for elements $a, b, c, d \in k[V]$. If f and g are equal on a nonempty open subset $U \subseteq V \setminus V(bd)$, then they are equal on $V \setminus V(bd)$ and they are equal as elements of $k(V)$.

(6) (a) Let D be a domain with field of fractions K. Let $f_1 = \frac{a_1}{b_1}, \ldots, f_r = \frac{a_r}{b_r} \in K$ where $a_1, \ldots, a_r \in D$ and $b_1, \ldots, b_r \in D \setminus \{0\}$ for all i. If $f_i = f_j$ in K for all i and j and if $(b_1, \ldots, b_r) = (1)$ in D, prove that there is an element $c \in D$ such that $c = f_i$ in K for all i.

(b) Let $V \subseteq \mathbf{A}^m(k)$ be an irreducible algebraic variety with k algebraically closed. Assume we have rational maps $\phi_i : V - - \to \mathbf{A}^1(k)$, $i = 1, \ldots, r$, where each ϕ_i is defined by $f_i \in k(V)$, where $f_i = a_i/b_i$ with $a_i, b_i \in k[V]$, and hence $\operatorname{ind}(\phi_i) = V(b_i)$. If $\cap_i \operatorname{ind}(\phi_i) = \varnothing$ and $\phi_i = \phi_j$ on $(V \setminus \operatorname{ind}(\phi_i)) \cap (V \setminus \operatorname{ind}(\phi_i))$ for all i and j, show there is a morphism $\phi : V \to \mathbf{A}^1(k)$ such that ϕ restricted to $V \setminus \operatorname{ind}(\phi_i)$ is ϕ_i for each i.