M953, Homework 4, due Friday, February 22, 2013

Instructions: Do any three problems.

(1) Let k be a field and let $J_1, J_2 \subseteq k[x_1, \ldots, x_m]$ be ideals.
   (a) Show that $V(J_1) = V(J_2)$ in $\mathbf{A}^m(k)$ if $\sqrt{J_1} = \sqrt{J_2}$.
   (b) If k is algebraically closed, show that $\sqrt{J_1} = \sqrt{J_2}$ if $V(J_1) = V(J_2)$ in $\mathbf{A}^m(k)$.
   (c) For any field k, show that $\sqrt{J_1} = \sqrt{J_2}$ if $V(J_1) = V(J_2)$, if for any ideal $J \subseteq k[x_1, \ldots, x_m]$ we take $V(J)$ to mean all K-points of $V(J)$ in $\mathbf{A}^m(K)$ for every field extension K of k. (Recall that a point $p \in \mathbf{A}^m(K)$ is a K-point of $V(J)$ if $f(p) = 0$ for all $f \in J$, where we regard J as a subset of $K[x_1, \ldots, x_m]$ via the inclusion $k[x_1, \ldots, x_m] \subseteq K[x_1, \ldots, x_m]$.) You may assume the fact from commutative algebra that $\sqrt{J'} \cap k[x_1, \ldots, x_m] = \sqrt{J}$, whenever $J \subseteq k[x_1, \ldots, x_m]$ is an ideal and J' is the ideal in $K[x_1, \ldots, x_m]$ generated by J.

(2) Let k be a field and let $S \subseteq \mathbf{A}^m(k)$ be any subset. Show that the closure of S in the Zariski topology is $V(I(S))$. (Recall that the closure of a set S is the intersection of all closed sets containing S.)

(3) Let $S \subset \mathbf{A}^2(\mathbf{R})$ be the set (a, b) with $ab = 1$ and $a > 0$. Show that the Zariski closure of S is $V(xy - 1)$, where $\mathbf{R}$ is the field of reals and we take the coordinate ring of $\mathbf{A}^2(\mathbf{R})$ to be $\mathbf{R}[x, y]$.

(4) Let $C = V(y - x^2) \subset \mathbf{A}^2(k)$, where $k = \mathbf{Q}$ is the field of rationals. Let K be the reals. Thus $\pi = 3.14159265\ldots \in K$. Show that $p = (\pi, \pi^2)$ is a K-point of C, and that $f(p) = 0$ for $f \in k[x, y]$ if and only if $g(x, y) = y - x^2$ divides f. Conclude that $J \cap k[x, y] = (y - x^2)$, where the intersection takes place in $K[x, y]$ and $J \subset K[x, y]$ is the ideal generated by $x - \pi$ and $y - \pi^2$. (You may assume that π is transcendental over k. Hint: apply Problem 5 on Homework 1.)

(5) Let k be a field and let $C_1, \ldots, C_r$ and $D_1, \ldots, D_s$ be irreducible algebraic varieties in $\mathbf{A}^m(k)$. Assume that $C_i \subseteq C_j$ implies $i = j$ and $D_i \subseteq D_j$ implies $i = j$ (i.e, none of the C's contains any of the others, and likewise for the D's). If $C_1 \cup \cdots \cup C_r = D_1 \cup \cdots \cup D_s$, show that $r = s$ and, after reordering if need be, that $C_i = D_i$ for each i.

(6) Let $C = V(J) \subset \mathbf{A^3}$ for the ideal $J = (xy^2 - xy, xyz - xz, y^3 - y^2, y^2z - yz, y^2z - y^2 - yz + y, yz^2 - yz - z^2 + z) \subset k[x, y, z]$, where k is an algebraically closed field. Find the irreducible components of C.