M953, Homework 3, due Friday, February 8, 2013

Instructions: Do any three problems.

(1) Let I and J be ideals in $k[x_1, \ldots, x_n]$, k a field. Show that $\mathrm{LT}(I \cap J) \subseteq \mathrm{LT}(I) \cap \mathrm{LT}(J)$ holds for all monomial orders, but that equality can fail.

(2) Let I and J be ideals in $k[x_1, \ldots, x_n]$, k a field. Show that $\mathrm{LT}(IJ) \supseteq \mathrm{LT}(I)\,\mathrm{LT}(J)$ holds for all monomial orders, but that equality can fail. (Hint: consider lexicographical ordering for $I = J \subset k[x, y]$ for $I = (x(x-1), y(y-1), xy)$ and show that $xy(x + y - 1) \in I^2$.)

(3) Let's collect some data. With an ideal of your own choice, compare the efficiency of using the lexicographic order versus graded reverse lexicographic order to compute a Gröbner basis. Here's a sample run, using Macaulay2 on math-compute-1.unl.edu. Notice for this example that graded revlex looks more efficient than lex.

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[math-compute-1 ~]$ M2
Macaulay2, version 1.3.1
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
PrimaryDecomposition, ReesAlgebra, SchurRings, TangentCone
i1 : gbTrace = 3
o1 = 3
i2 : S = QQ[a..c, MonomialOrder => GRevLex => {1,2,3}];
   -- registering polynomial ring 4 at 0x20f6500
i3 : I=ideal(c-a^5,b-a^3);
o3 : Ideal of S
i4 : g = gb I
   -- registering gb 1 at 0x248f8c0
   -- [gb]{3}(1)m{5}(1)m{6}(1)m{8}(1)o
   -- number of (nonminimal) gb elements = 3
   -- number of monomials                = 6
o4 = GroebnerBasis[status: done; S-pairs encountered up to degree 7]
o4 : GroebnerBasis
i5 : gens g
o5 = | a3-b b2-ac a2b-c |
i6 : R=QQ[x..z, MonomialOrder => Lex];
   -- registering polynomial ring 5 at 0x20f6200
i7 : J=ideal(z-x^5,y-x^4);
o7 : Ideal of R
i8 : h = gb J
   -- registering gb 2 at 0x248f700
   -- [gb]{4}(1)m{5}(1)m{8}(1)m{9}(2)om{10}(2)mo{11}(2)mo{12}(1)o
   -- number of (nonminimal) gb elements = 6
   -- number of monomials                = 12
o8 = GroebnerBasis[status: done; S-pairs encountered up to degree 11]
o8 : GroebnerBasis
i9 : gens h
o9 = | y5-z4 xz3-y4 xy-z x2z2-y3 x3z-y2 x4-y |
i10 : quit
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(4) Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Let $1 \le s \le n$, and let $J = I \cap k[x_s, \ldots, x_n]$. Let $f_1, \ldots, f_r$ be a Gröbner basis for I with respect to lex ordering. Assume that $\mathrm{LM}(f_1) > \cdots > \mathrm{LM}(f_r)$. Let j be the least index such that $\mathrm{LM}(f_j) \in k[x_s, \ldots, x_n]$. Show that $J = (f_j, \ldots, f_r)$.

(5) Let $I, J \subseteq k[x_1, \ldots, x_n]$ be ideals. Introduce a new variable t and consider

$$L = tI^* + (1-t)J^* \subseteq k[t, x_1, \ldots, x_n],$$

where I^* and J^* are the extended ideals $I^* = Ik[t, x_1, \ldots, x_n]$ and $J^* = Jk[t, x_1, \ldots, x_n]$. Show that $I \cap J = L \cap k[x_1, \ldots, x_n]$. (This result, together with Problem 4, gives a way to compute intersections of ideals using Gröbner bases.)