M953, Homework 2, due Friday, January 25, 2013

Instructions: Do any three problems.

(1) Let $p = (0, \dots, 0) \in k^n$ where k is any field (you may assume k is the field $\mathbf{C}$ of complex numbers if you like). Let $m \geq 0$ be an integer, and let I be the ideal of all polynomials that vanish at p to order at least m.

(a) Prove that I is a monomial ideal.

(b) Let I_t be the k-vector space span of all polynomials in I of degree at most t. Prove that $\dim_k I_t$ is 0 if $t < m$ and $\dim_k I_t = \binom{t+n}{n} - \binom{n+m-1}{n}$ if $t \geq m$. (Thus vanishing at p to order at least m imposes independent conditions on polynomials of degree t if and only if $t \geq m$.)

(2) Let $n \geq 1$ be an integer such that either n is a square or $n \geq 10$ and let $p_1, \dots, p_n$ be distinct points in k^2 for an algebraically closed field k. Let m be an integer $m \geq 0$ and let I be the ideal of all polynomials that vanish to order at least m at each point p_i. A consequence of the SHGH Conjecture (named after B. Segre, B. Harbourne, A. Gimigliano and A. Hirschowitz, each of whom posed versions of the conjecture, respectively in 1960, 1986, 1987 and 1989) is that if the points p_i are "sufficiently general," then $\dim_k I_t = \max(\binom{t+2}{2} - n\binom{m+1}{2}, 0)$. Show that $\dim_k I_t = \max(\binom{t+2}{2} - n\binom{m+1}{2}, 0)$ can fail for $n = 2$. (Hint: consider $m = 2$.)

(3) Let p_1, p_2 be distinct points in k^n for a field k with $n \geq 2$ an integer. Let m_1 and m_2 be nonnegative integers. Let $I \subseteq k[x_1, \dots, x_n]$ be the ideal of all polynomials vanishing to order at least m_i at p_i, for $i = 1, 2$. Show that the least degree δ of a polynomial $f \in I \setminus \{0\}$ is $m = \max(m_1, m_2)$.

(4) Consider the polynomial ring $k[x_1, \dots, x_n]$ over a field k with $n > 1$. Define a total ordering $\underset{\text{al}}{>}$ on monomials by setting $x_1^{a_1} \cdots x_n^{a_n} \underset{\text{al}}{>} x_1^{b_1} \cdots x_n^{b_n}$ if $x_1^{a_1} \cdots x_n^{a_n} \neq x_1^{b_1} \cdots x_n^{b_n}$ and for the least i such that $a_i \neq b_i$, either $a_i > b_i$ and $b_{i+1} + \cdots + b_n > 0$, or $a_i < b_i$ and $a_{i+1} + \cdots + a_n = 0$. (This is just the alphabetical ordering, where monomials are words and the letters in order are $x_1, \dots, x_n$.)

(a) Show by example that this ordering is not well ordered, and that there exist monomials M_1, M_2, M_3 such that $M_1 \underset{\text{al}}{>} M_2$ but $M_2 M_3 \underset{\text{al}}{>} M_1 M_3$.

(b) Show that reverse alphabetical ordering (denote this by $\underset{\text{ral}}{>}$, or as in class by $\underset{\text{la}}{>}$, so $M_1 \underset{\text{ral}}{>} M_2$ iff $M_1 \underset{\text{la}}{>} M_2$ iff $M_2 \underset{\text{al}}{>} M_1$) also suffers from the same two defects.

(5) If $M_1, M_2, M_3 \in k[x_1, \dots, x_n]$ (where k is a field and $n > 1$) are monomials such that $M_1 \underset{\text{al}}{>} M_2$ and $\deg(M_1) = \deg(M_2)$, show that $M_1 M_3 \underset{\text{al}}{>} M_2 M_3$.