Homework 2: Math 953 Spring 2005

Due January 28, 2005

(1) Let V be a finite dimensional vector space over a field k. Let $\mathcal{C}$ be the category of subspaces of V, where arrows are arbitrary linear transformations. Let $\mathcal{D}$ be the category of subspaces of V^{**}, where arrows are arbitrary linear transformations. Show that these are isomorphic categories.

(2) Let $\mathcal{C}$ be the category of finite dimensional real vector spaces, where arrows are arbitrary linear transformations. Let $\mathcal{D}$ be the category whose objects are the standard real vector spaces $\mathbf{R}^n$, $n \geq 0$, where arrows are arbitrary linear transformations. Show that these categories are not isomorphic but are equivalent.

(3) Let R be an integral domain. For any prime ideal $p \subset R$, let R_p denote the localization $S_p^{-1}R$, where $S_p = R - p$. We always have $R_p \subset R_{(0)}$; $R_{(0)}$ is the field of fractions of R. Thus it makes sense to take the intersection $\cap_{m \in M} R_m$, where M is the set of maximal ideals of R. Show that $\cap_{m \in M} R_m = R$.

(4) Let X be an affine variety contained in $\mathbf{A}_k^n$, where k is algebraically closed. Let $R = k[X]$ be the coordinate ring $k[\mathbf{A}_k^n]/I(X)$ of X. The sheaf $\mathcal{O}_X$ of regular functions on X is defined, for any (Zariski) open subset U of X, by $\mathcal{O}_X(U) = \cap_{p \in U} R_{I(p)}$. (Intuitively, a regular function f on U is a function which locally can be represented as a ratio of elements of the coordinate ring, as long as the representation in a neighborhood of each point $p \in U$ doesn't have a denominator that vanishes at p. You don't care what happens at points not in U.) And for any inclusion $U \subset V$ of open subsets of X, the functorial map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is just the obvious inclusion $\cap_{p \in V} R_{I(p)} \subset \cap_{p \in U} R_{I(p)}$. Also consider the presheaf $\mathcal{P}_X$ defined by $\mathcal{P}_X(U) = S_U^{-1}R$, where $S_U = \{f \in R : Z(f) \cap U = \emptyset\}$. The map $\mathcal{P}_X(V) \to \mathcal{P}_X(U)$ for an inclusion $U \subset V$, is again the obvious inclusion (note that $S_V \subset S_U$).

(a) Show that $\mathcal{P}_X(U) \subset \mathcal{O}_X(U)$, and that this defines a morphism $\mathcal{P}_X \to \mathcal{O}_X$ of presheaves.

(b) Show that $\mathcal{P}_X(U_f) = \mathcal{O}_X(U_f)$, where, for any $f \in R$, $U_f = X - Z(f)$. (Note: open sets of the form U_f form a basis for the Zariski topology on X, so (b) shows locally [i.e., on a basis of open sets] that $\mathcal{P}_X$ is the same as $\mathcal{O}_X$, and that therefore $\mathcal{O}_X$ is the sheafification of $\mathcal{P}_X$. In general, as the next problem shows, $\mathcal{P}_X \to \mathcal{O}_X$ need not be an isomorphism, so $\mathcal{P}_X$ need not be a sheaf.)

(5) Let $X = Z(xy - zw) \subset \mathbf{A}_k^4$ (assume k algebraically closed). For any $f \in k[x, y, z, w]$, let U_f be the open subset of $\mathbf{A}_k^4$ as defined above, and let $V_f = U_f \cap X$. Let $V = V_y \cap V_z$ and let $W = V_y \cup V_z$.

(a) Show that $S_W = k$. (Here's one way. First figure out what it means to be in S_W. Then define an appropriate injective homomorphism $R \to k[a, b, c]$, and avail yourself of unique factorization. What is going on here is that there is a polynomial map $F : \mathbf{A}_k^3 \to X$ which gives an isomorphism from a dense open subset of $\mathbf{A}_k^3$ to a dense open subset of X, inducing an injection $R \to k[a, b, c]$ of coordinate rings. Let $C = X - W$; since W is open, C is closed. For $f \in S_W$ to hold, we need $Z(f) \subset C$, or equivalently, we need $Z(f \circ F) \subset F^{-1}(C)$, which is to say we need $I(F^{-1}(C)) \subset \sqrt{(f \circ F)}$. But $\sqrt{(f \circ F)}$ is a principal ideal, so it is fairly small as long as f is not a nonzero constant. With the fact that you're working in a UFD, this gives you hope of explicitly showing $I(F^{-1}(C)) \subset \sqrt{(f \circ F)}$ does not hold unless $f \in k$, and hence that $S_W = k$.) Conclude that $\mathcal{P}_X(X) \to \mathcal{P}_X(W)$ is the identity.

(b) Show that x/z is not in the image of $\mathcal{P}_X(W) \to \mathcal{P}_X(V_z)$, but that $x/z \in \mathcal{P}_X(V_z)$ and $w/y \in \mathcal{P}_X(V_y)$, and that x/z under $\mathcal{P}_X(V_z) \to \mathcal{P}_X(V)$ has the same image as w/y under $\mathcal{P}_X(V_y) \to \mathcal{P}_X(V)$ (i.e., $x/z = w/y$ on V). Conclude that $\mathcal{P}_X$ fails to have the sheaf property; i.e., x/z is a function regular on V_z and w/y is regular on V_y, and they patch together to give a function regular on W, but this function is not contained in $\mathcal{P}_X(W)$.