M918: Algebraic Curves Problem Assignment 5, due Monday, November 23, 2009.

(Do the problems appropriate for your background or those for which you'd like feedback on.)

[1] Let A be an effective divisor on a smooth plane curve C and let B be a divisor on C. Show that $\mathcal{L}_C(B) \subseteq \mathcal{L}_C(A+B)$.

[2] Let $D \in \mathrm{Div}(C)$ be an effective divisor, where C is a smooth plane curve. Being a divisor we have $D = \sum_{q\in C} m_q q$, and being effective we have $m_q \geq 0$ for all q. The set of points $q \in C$ such that $m_q > 0$ is called the support of D. For any divisor A, let $|A| + p$ denote the set $\{E + p \in \mathrm{Div}(C) : E \in |A|\}$. Recall that we say $p \in C$ is a base point of $|D|$ if for each $\sum_{q\in C} m_q q \in |D|$ we have $m_p > 0$. Show that the following are equivalent:
(a) p is a base point of $|D|$; (b) $|D - p| + p = |D|$; (c) $\dim\mathcal{L}_C(D - p) = \dim\mathcal{L}_C(D)$.

[3] Let $D \in \mathrm{Div}(C)$ be an effective divisor, where C is a smooth plane curve of genus g_C. Prove that $|D|$ is base point free if $\deg(C) \geq 2g_C$.

Let C be a smooth plane curve over the complex numbers. Let $p \in C$. Let $K(C)^*$ denote the nonzero elements of the function field $K(C)$. Define a map $m_p : K(C)^* \to \mathbf{Z}$ to the integers by setting $m_p(f)$ to be the coefficient of p in the divisor $\mathrm{div}(f)$. You may assume that m_p is a group homomorphism; i.e., $m_p(fg) = m_p(f) + m_p(g)$ (this follows easily from our definition of $\mathrm{div}(f)$).

If D is an effective divisor and p is not in the support of D we can define a map $e_p : \mathcal{L}_C(D) \to \mathbf{C}$ by evaluation; i.e., $e_p(f) = f(p)$. This is well-defined since $f \in \mathcal{L}_C(D)$ implies f cannot have a pole at p.

[4] Let C be a smooth plane curve over the complex numbers. Let $p \in C$. Let $V_p = \cup_{n\in\mathbf{Z}}\mathcal{L}_C(np)$.

(a) Show that V_p is a complex vector subspace of $K(C)$.

(b) Show that V_p is infinite dimensional.

(c) Show that V_p is in fact a subring of $K(C)$.

(d) Let $q \in C$, $q \neq p$. Show that e_q defines a ring homomorphism $V_p \to \mathbf{C}$; conclude that $M_q = e_q^{-1}(0)$ is a maximal ideal of V_p.

Aside (for those with sufficient background): $U_p = C \setminus \{p\}$ is an affine open subset of C, which can be regarded as being $\mathrm{Spec}(V_p)$. The (closed) points of U_p are in bijective correspondence with the maximal ideals of V_p.

[5] Let C be a smooth plane curve over the complex numbers. Let $\mathcal{O}_p = \{0\} \cup \{0 \neq f \in K(C) : m_p(f) \geq 0\}$.

(a) Show that $\mathcal{O}_p$ is a complex vector subspace of $K(C)$.

(b) Show that $\mathcal{O}_p$ is a subring of $K(C)$.

(c) For every $0 \neq f \in K(C)$, show that either $f \in \mathcal{O}_p$ or $1/f \in \mathcal{O}_p$.

(d) Let $J_p = \{f \in K(C) : 1/f \notin V_p\}$. Show that J_p is the unique maximal ideal of $\mathcal{O}_p$.

(e) Let x be a linear form for a line in $\mathbf{P}^2$ through p that is not the tangent line, and let y be a linear form for a line in $\mathbf{P}^2$ not through p. Show that the equivalence class h of x/y, as an element of $K(C)$, is in J_p. In fact, for each $f \in J_p$ there is an $i > 0$ and an invertible $a \in \mathcal{O}_p$ (both depending on f) such that $f = ah^i$, in which case $m_p(f) = i$. (You may assume that $m_p(h) = 1$, but you can prove this if you want.)

(f) Let $f_1, f_2 \in K(C)^*$ such that $f_1 + f_2 \neq 0$. Show that $m_p(f_1 + f_2) \geq \min(m_p(f_1), m_p(f_2))$.

Aside (for those with sufficient background): $\mathcal{O}_p$ is a DVR (discrete valuation ring) with valuation m_p.