M918: Algebraic Curves Problem Assignment 4, due October 21, 2009.

(Do the problems appropriate for your background or those for which you'd like feedback on.)

[1] Over an algebraically closed field k of characteristic 2, show that there is a unique irreducible cuspidal cubic projective plane curve, given, up to change of coordinates, by $y^2z-x^3=0$ and that every smooth point is a flex point. If k has characteristic 3, however, show that there are up to change of coordinates exactly two irreducible cuspidal cubic projective plane curves, given by $y^2z-x^3=0$ or by $y^2z-x^3-x^2y=0$, and that every smooth point of the former is a flex and no smooth point of the latter is a flex.

[2] Let $C\subset \mathbf{P}^2$ be defined (over any algebraically closed field k) by an irreducible homogeneous polynomial F of degree 2 (i.e., C is a smooth conic). Show by appropriate choice of coordinates that F may be taken to be $yz-x^2$, regardless of the characteristic of k, but that there is a single point t common to every tangent line if and only if the characteristic is 2. (See Hartshorne's book, p. 311, for more discussion of such "strange" curves.)

[3] Let $f(x,y)\subset k[x,y]$ be an irreducible polynomial of degree 2, where k is any algebraically closed field. Let ρ be a solution $f(\rho)=0$. Show that there is a scalar $d\neq 0$ and an affine change of coordinates such that ρ and $f(x,y)/d$ become either $(0,0)$ and $v-u^2$, or $(1,1)$ and $uv-1$. I.e., show that there is an invertible matrix $\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)\in \mathrm{GL}_2(k)$ and constants $d\neq 0$, c and c' such that in terms of $u=ax+by+c$ and $v=a'x+b'y+c'$, f/d takes the form $v-u^2$ or $uv-1$, and ρ has coordinates $(0,0)$ or $(1,1)$, respectively. (Hint: make f homogeneous by introducing a variable z, but don't mess with z.)

[4] Let C be the curve defined in k^2 by an irreducible polynomial $f\in k[x,y]$ of degree 2, where k is any algebraically closed field, and let p be any point of C. Let $F\in k[x,y,z]$ be the homogeneous degree 2 polynomial obtained from f. Show that zF defines a cubic consisting of C together with the line $z=0$, and that our procedure for defining a group law on a cubic (taking p to be the identity) induces a group law on C. Show that the group obtained is isomorphic to $(k,+)$ if $F=0$ has a single root with $z=0$, and that the group is $(k^*,\cdot)$ if $F=0$ has two distinct roots with $z=0$.

[5] Let C be a smooth plane cubic curve over any algebraically closed field k. Assume for simplicity that C has a flex point p which we take to be the identity element of the group law on C. Let $F(C)$ be the free abelian group on the points of C (also known as the group of divisors on C). Let $\phi: F(C)\to (C,\oplus)\times \mathbf{Z}$ be the canonical homomorphism induced by $\phi(p_i)=(p_i,1)$. Show that $(C,\oplus)\times\mathbf{Z}$ is isomorphic to $F(C)/\ker(\phi)$, and that $\ker(\phi)$ is the subgroup Γ generated by elements of the form $a+b+c-d-e-f$, where a,b,c are collinear points of C and so are d,e,f.

Asides: (i) Elements of $F(C)$ which differ by an element of $\ker(\phi)$ are said to be linearly equivalent. Linear equivalence of divisors can be defined for any smooth projective curve C; the linear equivalence classes in $F(C)$ form a quotient group of $F(C)$, known as the divisor class group, $\mathrm{Cl}(C)$. It is also canonically isomorphic to what is known as the Picard group of C. Formally, the Picard group $\mathrm{Pic}(C)$ is the group of isomorphism classes of locally free rank 1 sheaves on C, with the group operation being tensor product.

(ii) The generalization of Problem [5] C is singular, but defined by a square-free polynomial, is that $F(C^*)/\Gamma\to (C^*,\oplus)\times\mathbf{Z}^r$ is an isomorphism, where C^* is the set of smooth points of C, and r is the number of irreducible components of C. Thus the group law on the cubic can be understood in terms of the group law on divisor classes induced by addition in the free abelian group on the smooth points.