M918: Algebraic Curves Problem Assignment 2, due Wednesday, September 16, 2009.

(Do the problems appropriate for your background or those for which you'd like feedback on.)

[0] Prove Euler's identity; i.e., if $F \in \mathbf{C}[x_0, \ldots, x_n]$ is homogeneous of degree $r > 0$, then $rF = \sum_i x_i F_i$, where $F_i = \partial F/\partial x_i$. (Keep in mind that with this notation sometimes a subscript is a subscript, but sometimes it's a differential operator; x_i is not $\partial x/\partial x_i$, for example.)

From one point of view, a change of coordinates keeps points the same but changes the coordinates, but you can also think of it as an automorphism which moves the points and leaves the coordinate system alone.

Let $M \in \mathrm{GL}_3(\mathbf{C})$. This defines an automorphism Φ_M of $\mathbf{C}^3$ (which also induces an automorphism of $\mathbf{P}^2$), defined by $\Phi_M : p = (a_0, a_1, a_2) \mapsto p' = (a'_0, a'_1, a'_2)$, where $M(a_0, a_1, a_2)^T = (a'_0, a'_1, a'_2)^T$. Regarding elements of $\mathbf{C}[x_0, x_1, x_2]$ as functions on $\mathbf{C}^3$, this also induces an automorphism $\Phi^*_M : \mathbf{C}[x_0, x_1, x_2] \to \mathbf{C}[x_0, x_1, x_2]$ defined by composition as $\Phi^*_M : F \mapsto F \circ \Phi_M$. You can if you like regard p' as the same point as p expressed in new coordinates, and likewise you can regard $\Phi^*_{M^{-1}}(F)$ as the same function as F expressed in terms of the new coordinates given by M (see Problem [2]).

[1] Show that $\gamma : M \mapsto \Phi^*_{M^{-1}}$ defines a group homomorphism $\gamma : \mathrm{GL}_3(\mathbf{C}) \to \mathrm{Aut}_{\mathbf{C}}(\mathbf{C}[x_0, x_1, x_2])$ from invertible matrices to $\mathbf{C}$-automorphisms of the coordinate ring. (A $\mathbf{C}$-automorphism is an automorphism which is the identity on constants.)

[2] Let $G = \Phi^*_{M^{-1}}(F)$. Show that $G(p') = F(p)$ for all $p \in \mathbf{C}^3$ (i.e., the values of functions are invariant under change of coordinates). Conclude that $\Phi_M(V(F)) = V(G)$ (i.e., a 0-locus is invariant under changes of coordinates).

[3] Define the gradient ∇F of F to be the vector (F_0, F_1, F_2). If $G = \Phi^*_{M^{-1}}(F)$ and $p' = \Phi_M(p)$, show that

$$\Phi_{M^T}((\nabla G)(p')) = (\nabla F)^T(p).$$

Conclude that $(\nabla F)(p) = (0, 0, 0)$ if and only if $(\nabla G)(p') = (0, 0, 0)$ (i.e., the singular locus of a curve is invariant under change of coordinates).

[4] Define the Hessian $H(F)$ of F to be the determinant of the matrix whose entries are $F_{ij} = \partial^2 F/\partial x_i \partial x_j$. If $G = \Phi^*_{M^{-1}}(F)$ and $p' = \Phi_M(p)$, show that $(\det(M))^2 H(G)(p') = H(F)(p)$. Conclude that $H(G)(p') = 0$ if and only if $H(F)(p) = 0$ (i.e., the 0-locus of the Hessian is invariant under change of coordinates).