M918: Algebraic Curves Problem Assignment 1, due Friday, September 4, 2009.

(Do the problems appropriate for your background or those for which you'd like feedback on. You may assume $k = \mathbf{C}$ if you prefer.)

[0] Let k be a field and let $a_1, \ldots, a_n \in k$. Then $M = (x_1 - a_1, \ldots, x_n - a_n) \subset k[x_1, \ldots, x_n]$ is a maximal ideal.

[1] Let $\mathcal{S} \subseteq k[x_1, \ldots, x_n$, where k is a field and let $\bar{a} = (a_1, \ldots, a_n) \in k^n$. Show $f(\bar{a}) = 0$ for all $f \in \mathcal{S}$ if and only if $I(\mathcal{S}) \subseteq (x_1 - a_1, \ldots, x_n - a_n)$.

[2] Let $h \in k[x, y]$ be non-constant, where k is an algebraically closed field. Show that $V(h)$ is infinite.

[3] Let $f, g \in k[x, y]$ where k is an algebraically closed field. If f and g have a non-constant common factor, show that $V(f, g)$ is infinite.

For the next problem, recall that the radical $\sqrt{I}$ of an ideal I in a commutative ring R is the intersection of the prime ideals $P \subset R$ which contain I (where the empty intersection is by convention R itself), and that when R is a polynomial ring $k[x_1, \ldots, x_n]$ over a field k, $\sqrt{I}$ is the intersection of finitely many of the primes which contain I. Also recall for $R = k[x, y]$, in the case that k is algebraically closed, that every prime ideal is either principal or of the form $(x - b, y - c)$ for constants $b, c \in k$.

[4] Let $f, g \in k[x, y]$ where k is an algebraically closed field. If f and g have no non-constant common factor, show that $V(f, g)$ is finite.

For the next problem, it is helpful to note that if $f(x, y) \in k[x, y]$ is a polynomial of degree r with coefficients in a field k, then $z^r f(x/z, y/z) \in k[x, y, z]$ is a homogeneous polynomial of degree r, and that if $F(x, y, z)$ is non-zero and homogeneous of degree t and r is the degree of $F(x, y, 1)$, then $t \geq r$ and $F(x, y, z) = z^r f(x/z, y/z) z^{t-r}$.

[5] If k is a field and if $F, G \in k[x, y, z]$ are homogeneous with no common factor of positive degree, then $F(x, y, 1)$ and $G(x, y, 1)$ also have no common factor of positive degree.