M902-2009 Assignment 7: Due Wednesday April 8

Instructions: Do any three of the following problems.

(1) Prove I. S. Cohen's theorem: If A is a commutative ring (with $1 \neq 0$) such that every prime ideal is finitely generated, then A is Noetherian. (You may assume without proof that any ideal which is maximal among the set S of all non-finitely generated ideals of A is prime. This is Proposition 2.4, p. 379, of Hungerford.)

(2) Let A be a commutative ring (with $1 \neq 0$). Let $I \subseteq A$ be an ideal. Assume that A/I is a projective A-module. Show that I is also projective and that $I = (a)$ for some element $a \in A$.

(3) Let A be a commutative ring (with $1 \neq 0$) and let M be a flat A-module. Show that the following are equivalent: M is faithfully flat; $mM \subsetneq M$ for every maximal ideal $m \subset A$; and $B \otimes_A M \neq 0$ whenever B is a nonzero A-module.

(4) Let A be a commutative ring (with $1 \neq 0$) and let P be a projective A-module. Say that P is a faithfully projective A-module if and only if any sequence $0 \to B \to C \to D \to 0$ of A-modules is short exact exactly when the result of applying $\mathrm{Hom}_A(P, \cdot)$ is short exact. Show that the following are equivalent: $\mathrm{Hom}_A(P, B) \neq 0$ for every A-module $B \neq 0$; P is faithfully projective; $\mathrm{Hom}(P, A/I) \neq 0$ for every proper ideal $I \subset A$; and $\mathrm{Hom}(P, A/m) \neq 0$ for every maximal ideal $m \subset A$.

(5) Let A be a commutative ring (with $1 \neq 0$). Let P be a projective A-module. This problem shows that being faithfully projective is the same as being projective and faithfully flat.

  (a) If P is faithfully projective, show that if B is an A-module such that $B \otimes_A P = 0$, then $B = 0$; conclude that P is faithfully flat.

  (b) If P is faithfully flat, show that P is faithfully projective.

(6) Let A be a commutative ring (with $1 \neq 0$). Give an exact sequence $0 \to B \to C \to D \to 0$ of A-modules such that $\mathrm{ass}(C) \subsetneq \mathrm{ass}(B) \cup \mathrm{ass}(D)$.