M901, Exam 2: Friday, December 2, 2011

Instructions: Do any three problems.

(1) Prove that S_4 is solvable but not nilpotent.

(2) Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of quaternion units (i.e., the group of order 8 where ± 1 are in the center, $(-1)^2 = 1$, $i^2 = j^2 = k^2 = -1$, $-1 * i = -i$, $-1 * j = -j$, $-1 * k = -k$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$). Show that Q is not isomorphic to a semi-direct product of two groups of orders less than 8.

(3) Assume that every algebraic extension F of E is separable. Show that E is perfect.

(4) Let K have $\text{char}(K) = p > 0$. Let L be a finite extension field of K such that p does not divide $[L : K]$. Show that L is a separable extension of K.

(5) Let a be algebraic over a field k. Assume $Irr(a, k, x)$ has odd degree. Show that $k(a) = k(a^2)$.

(6) Let E be an algebraically closed field. Let k be the prime subfield of E (i.e., the smallest subfield containing 1). Prove that $[E : k]$ is infinite. [You may assume that a prime field of characteristic 0 is isomorphic to the rationals.]