Given a finite dimensional real vector space V with a real symmetric bilinear form $\langle\ ,\ \rangle$, here are algorithms for finding a basis for the space which is orthogonal with respect to $\langle\ ,\ \rangle$.

Positive-definite case: The Gram-Schmidt algorithm applies when $\langle\ ,\ \rangle$ is positive-definite. Suppose S is a finite set that spans a subspace V of a real vector space W with a positive definite real symmetric form $\langle\ ,\ \rangle$. Last semester I gave a version of the Gram-Schmidt algorithm that can be used to obtain from S an orthogonal basis B for V. (The algorithm in Artin's book assumes S is a basis of V and that $W = V$.) To make it easier to see how to generalize Gram-Schmidt to handle non-positive definite forms, I'll describe Gram-Schmidt using slightly different notation than last semester. If $S = \{s_1, \ldots, s_r\}$, the procedure from last semester gives rise to vectors $\{u_1, \ldots, u_r\}$, and then $B = \{u_i : u_i \neq 0\}$. Our procedure for obtaining the u_i involved constants c_i, where $c_i = 1$ if $\langle u_i, u_i\rangle = 0$, and otherwise $c_i = \langle u_i, u_i\rangle$. Here's the procedure, step-by-step:

(1) Let $S = \{s_1, \ldots, s_r\}$, and let $B = \emptyset$, to start.

(2) Let v be the first element of S, and take it out of S (i.e., redefine S to be $S - \{v\}$). Define v' to be

$$v' = v - \sum_{u\in B}\langle v, u\rangle u/c_u$$

and let $c_{v'} = \langle v', v'\rangle$. If $c_{v'} \neq 0$, then add v' to B (i.e., redefine B to be $B \cup \{v'\}$). Otherwise leave B alone.

(3) Keep repeating step 2 until $S = \emptyset$.

The set B you end up with is a basis for $\mathrm{Span}(\{s_1, \ldots, s_r\})$ orthogonal with respect to $\langle\ ,\ \rangle$, but not necessarily orthonormal. You need to divide each u in B by $\sqrt{c_u}$ to get an orthonormal basis.

A general case algorithm: In general (i.e., if $\langle\ ,\ \rangle$ is not necessarily positive definite), a modified version of Gram-Schmidt can be used. Here are the steps:

(1) Pick a basis B for the nullspace N of $\langle\ ,\ \rangle$. (So at the start of *this* algorithm B is not empty.)

(2) Extend B to a basis $B \cup S$ of V. Let $S' = \emptyset$.

(3) Let v be the first element of S, and take it out of S (i.e., redefine S to be $S - \{v\}$). Define v' to be

$$v' = v - \sum_{u\in B}\langle v, u\rangle u/c_u$$

and let $c_{v'} = \langle v', v'\rangle$. If $c_{v'} = 0$, then put v' into S' (i.e., redefine S' to be $S' \cup \{v'\}$), but leave B alone. If $c_{v'} \neq 0$, then add v' to B (i.e., redefine B to be $B \cup \{v'\}$), put the elements of S' back into S and reset S' to be empty (i.e., redefine S to be $S \cup S'$ and then redefine S' to be empty).

(4) Keep repeating step 3. Eventually, either S and S' will both be empty (in which case you're done, and B is a basis for V orthogonal with respect to $\langle\ ,\ \rangle$), or S will be empty but S' will not be. In this second situation, every element of S' is orthogonal to every element of B, but $c_x = 0$ for every $x \in S'$. If S' has only a single element, move it into B (i.e., redefine B to be $B \cup S'$); then B is an orthogonal basis and you're done. If S' has two or more elements, let $u = v + w$, where v is the first element of S' and w is any element of S' such that $\langle v, w\rangle \neq 0$ (see the note (*) for why such a w exists), then move u into B, move all but the first element v of S' back into S, reset S' to be empty, and go back to repeating step 3 (and step 4 if it ever happens that S becomes empty but S' is not). The end result of applying steps 3 and 4 is that S gets smaller and S' is empty, so eventually both S and S' will be empty, and you're done: the B you end up with is an orthogonal basis for V. (Note *: $c_u = \langle u, u\rangle$ simplifies to $2\langle v, w\rangle$, hence is nonzero. But how do we know that an appropriate w exists? Recall N is in the span of B and $B \cup S' \cup S$ is always a basis for V, so no element of S' can be in N. If no w exists, this means v is orthogonal to every element of S', but v, being in S', is also orthogonal to every element of B, which would mean v is in N.)

The nonexplicit algorithm from last semester for the general case: I gave a somewhat nonexplicit routine for the general case last semester. It's similar to but conceptually a bit simpler then the general case algorithm above; however, it's not as efficient. The main difference is that in the algorithm above we try to choose elements of S one at a time, adjusting each choice to make it orthogonal to what is already in B, then (as long as $c \neq 0$ for our adjusted choice) we add it to B and delete our choice from S. Only when $c = 0$ for every choice do we do something different, which involves extra work. The algorithm from last semester does this extra work every time an element is to be included in B. Assume $\dim V = n$ and let N be the nullspace of $\langle\ ,\ \rangle$.

(1) If $n = 1$ or $N = V$, any basis B is orthogonal with respect to $\langle\ ,\ \rangle$.

(2) If $n > 1$ and N is a proper subspace of V, then:

  (i) pick $w \in V$ such that $\langle w, w\rangle \neq 0$; such a w exists by Proposition 2.2 on p. 243. Let $W = \mathrm{Span}(\{w\})$ and note that $\dim W^\perp = n - 1$.

  (ii) pick a basis B' of $W^\perp$ orthogonal with respect to $\langle\ ,\ \rangle$.

(3) Then $B = B' \cup \{w\}$ is a basis of V which is orthogonal with respect to $\langle\ ,\ \rangle$. Note that (2)(ii) is iterative: if either $\dim W^\perp = 1$ or the nullspace of $W^\perp$ is all of $W^\perp$, then (as in (1)) any basis B' of $W^\perp$ is orthogonal. If neither condition obtains, we repeat step (2) (i.e., we pick a new vector w, this time in $W^\perp$, etc) and get a new (and smaller) $W^\perp$. Because the dimension of $W^\perp$ keeps getting smaller, eventually step (1) will apply; our sequence of choices of w together with any basis of the final $W^\perp$ gives an orthogonal basis for V.