Homework 8, due Tuesday, November 27, 2012

Do any 4 of the 6 problems. Each problem is worth 25 points. Solutions will be graded for correctness, clarity and style.

(1) Let X be a topological space and let Y be a set. Let $f : X \to Y$ be a map, not necessarily surjective. Let $\mathcal{T}_Y = \{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$. Show that $\mathcal{T}_Y$ defines a topology on Y. (In class we used this construction mainly when f is surjective.)

Solution: Since $f^{-1}(\varnothing) = \varnothing$ is open in X, we see that $\varnothing \in \mathcal{T}_Y$. Since $X = f^{-1}(Y)$ is open in X, we see that $Y \in \mathcal{T}_Y$. If $V_1, V_2 \in \mathcal{T}_Y$, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are open in X, so $f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2)$ is open, hence $V_1 \cap V_2 \in \mathcal{T}_Y$. Finally, if $V_i \in \mathcal{T}_Y$ for all $i \in I$ for some index set I, then $f^{-1}(V_i)$ is open in X for all $i \in I$, hence $f^{-1}(\cup_{i\in I} V_i) = \cup_{i\in I} f^{-1}(V_i)$ is open in X, so $\cup_{i\in I} V_i \in \mathcal{T}_Y$. Thus

(2) Let X be a topological space and let Y be a set. Let $f : X \to Y$ be a map such that $f(X)$ is a single point $p \in Y$. Let $\mathcal{T}_Y = \{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$. Show that $\mathcal{T}_Y$ is the discrete topology on Y.

Solution: Since we know $\mathcal{T}_Y$ is a topology by Problem 1, it is enough to show that $\{y\}$ is open for each point $y \in Y$. If $y = p$, then $f^{-1}(\{y\}) = X$, so $\{y\}$ is open. If $y \neq p$, then $f^{-1}(\{y\}) = \varnothing$, so $\{y\}$ is open.

(3) Let σ^k be the standard k-simplex, so $\sigma^k = \{(x_1, \ldots, x_{k+1}) \in \mathbf{R}^{k+1} : x_1 \geq 0, \ldots, x_{k+1} \geq 0, x_1 + \cdots + x_{k+1} = 1\}$. Let τ^k be a k-simplex in $\mathbf{R}^N$, so $\tau^k = \langle v_1, \ldots, v_{k+1}\rangle$ where the points $v_i \in \mathbf{R}^N$ are geometrically independent. Define a map $f : \sigma^k \to \tau^k$ by $f((x_1, \ldots, x_{k+1})) = \sum_i x_i v_i$.

(a) Show that f is bijective.

(b) Assuming that f is continuous, show that f is a homeomorphism. This shows that all k-simplices are homeomorphic to the standard k-simplex and hence to each other. [Aside: $h : \mathbf{R}^{k+1} \to \mathbf{R}^N$ defined by $f((a_1, \ldots, a_{k+1})) = \sum_i a_i v_i$ is a linear transformation. It is known and not hard to show that linear transformations are continuous under the standard topologies, basically because a linear transformation just involves addition and multiplication of the variables by constants, and these are continuous. Since f is the restriction of h to σ^k, this means that f is indeed continuous.]

Solution: (a) First, f is surjective since every point $b \in \tau^k$ satisfies $b = a_1 v_1 + \cdots + a_{k+1} v_{k+1}$ for some real numbers $a_i \geq 0$ with $a_1 + \cdots + a_{k+1} = 1$. Thus $(a_1, \ldots, a_{k+1}) \in \sigma^k$ and $f((a_1, \ldots, a_{k+1})) = b$. For injectivity, we assume that $f((a_1, \ldots, a_{k+1})) = f((a'_1, \ldots, a'_{k+1}))$ for points $(a_1, \ldots, a_{k+1}), (a'_1, \ldots, a'_{k+1}) \in \sigma^k$ and we want to show that the points are the same. Thus $a_1 v_1 + \cdots + a_{k+1} v_{k+1} = a'_1 v_1 + \cdots + a'_{k+1} v_{k+1}$, so $(a_1 - a'_1)v_1 + \cdots + (a_{k+1} - a'_{k+1})v_{k+1} = 0$, hence $(a_1 - a'_1)(v_1 - v_1) + (a_2 - a'_2)(v_2 - v_1) + \cdots + (a_{k+1} - a'_{k+1})(v_{k+1} - v_1) = (a_1 - a'_1)v_1 + (a_2 - a'_2)v_2 + \cdots + (a_{k+1} - a'_{k+1})v_{k+1} - ((a_1 - a'_1) + (a_2 - a'_2) + \cdots + (a_{k+1} - a'_{k+1}))v_1 = (a_1 - a'_1)v_1 + (a_2 - a'_2)v_2 + \cdots + (a_{k+1} - a'_{k+1})v_{k+1} - (1 - 1)v_1 = (a_1 - a'_1)v_1 + (a_2 - a'_2)v_2 + \cdots + (a_{k+1} - a'_{k+1})v_{k+1} = 0$. But the vectors $v_2 - v_1, \ldots, v_{k+1} - v_1$ are linearly independent so $(a_1 - a'_1) = \cdots = (a_{k+1} - a'_{k+1}) = 0$, hence $a_i = a'_i$ for all i, so $(a_1, \ldots, a_{k+1}) = (a'_1, \ldots, a'_{k+1})$, hence f is injective.

(b) The function $g_i((x_1, \ldots, x_{k+1})) = x_i$ is the projection map $\pi_i : \mathbf{R}^{k+1} \to \mathbf{R}$, and this is continuous since the standard topology on $\mathbf{R}^{k+1}$ is the product topology, and the projection maps are continuous for the product topology. But $\sigma^k = D \cap$

$(\cap_{i=1}^{k+1} g_i^{-1}(C_i))$ where C_i is the closed set $[0,\infty)$ and D is the closed set $\lambda^{-1}(\{1\})$, where $\lambda : \mathbf{R}^{k+1} \to \mathbf{R}$ is the continuous function $\lambda((x_1,\cdots,x_{k+1})) = x_1 + \cdots + x_{k+1}$. Thus σ^k is closed. But σ^k is also bounded so it is compact. Since τ^k is Hausdorff (being a subset of a Hausdorff space with the subspace topology), the bijective continuous function f must be a homeomorphism.

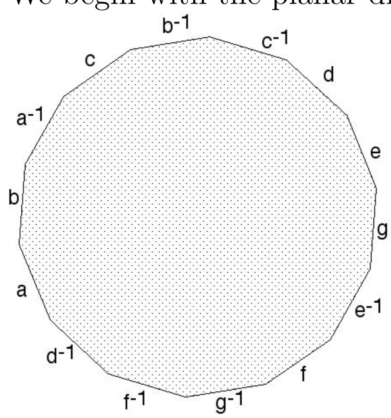
(4) Let $0 \le r \le k \le K$ where r, k and K are integers. Give a formula for the number of k-simplices contained in a K-simplex which contain a given r-simplex. [For example, if $r = 0$, $k = 1$ and $K = 3$, this is asking how many edges of a tetrahedron contain a given vertex of the tetrahedron, so your formula should give 3.]

Solution: Let the given r-simplex be $\alpha^r = \langle v_0, \ldots, v_r\rangle$. Let the K-simplex be $\sigma^K = \langle v_0, \ldots, v_K\rangle$. Thus a k-simplex β^k that contains α^r and is contained in σ^K must be of the form $\langle v_0, \ldots, v_r, v_{i_{r+1}}, \ldots, v_{i_k}\rangle$, where $v_{i_{r+1}}, \ldots, v_{i_k}$ are distinct elements of $\{v_{r+1}, \ldots, v_K\}$. I.e., the number of such k-simplices is the number of ways to choose $k-r$ things from a set of $K-r$ things, hence $\binom{K-r}{k-r}$.

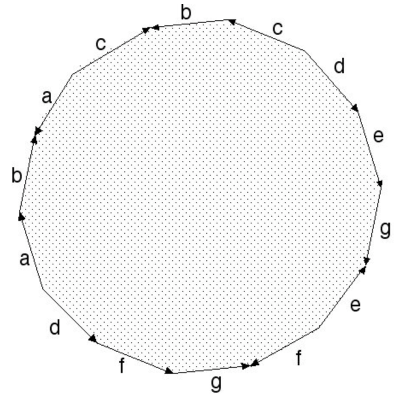
(5) Consider a 14-gon. Go around the circumference of the 14-gon, labeling the edges in turn as follows: a, b, a^{-1}, c, b^{-1}, c^{-1}, d, e, g, e^{-1}, f, g^{-1}, f^{-1}, d^{-1}. This gives a planar diagram, with labels and exponents, for a multi-holed torus. (The planar diagram can also be shown with labels and arrows as in the diagram below at right. Think of a as specifying an arrow on the corresponding edge of the 14-gon pointing in the direction in which you're going around the 14-gon. Think of a^{-1} as specifying an arrow on the corresponding edge of the 14-gon but pointing in the direction opposite to which you're going around the 14-gon.) Determine the number of holes of the torus you get by making the identifications specified by the labels.

Solution: This is a planar diagram for a two holed torus. We do this three ways: first using the Euler characteristic, then carrying out the gluings in an ad hoc way to see what we get, then using a disassembly and reassembly method.

We begin with the planar diagram using labels and exponents:

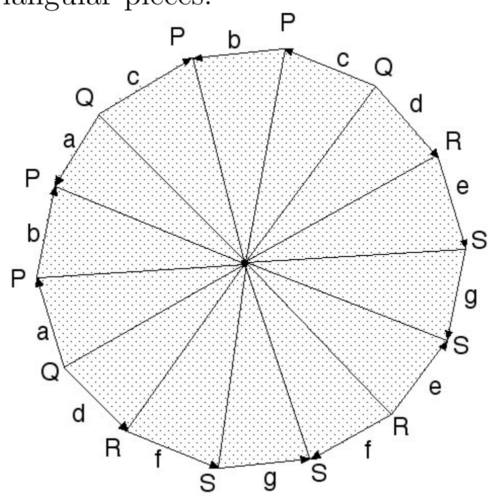


Here is the same diagram using arrows instead of exponents:



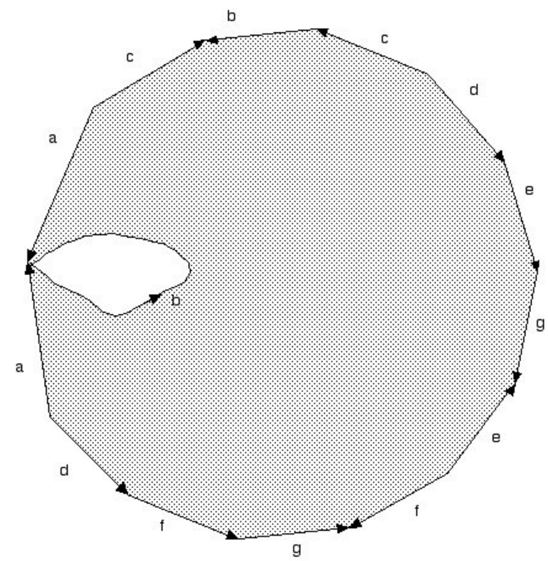
The symbolic representation for the planar diagram is $aba^{-1}cb^{-1}c^{-1}dege^{-1}fg^{-1}f^{-1}d^{-1}$. Using the usual rules of algebra, this simplifies to 1, so the 2-manifold X it represents is oriented. For an oriented compact 2-manifold X the Euler characteristic $\chi(X)$ is $2-2g$ where X is a g-holed torus. Once we know $\chi(X)$ we can find g. (If X were not orientable, then we would have $\chi(X) = 2 - i$ where X is a connected sum of i copies of the real projective plane.)

To compute the Euler characteristic, we need to break the planar diagram into triangular pieces:

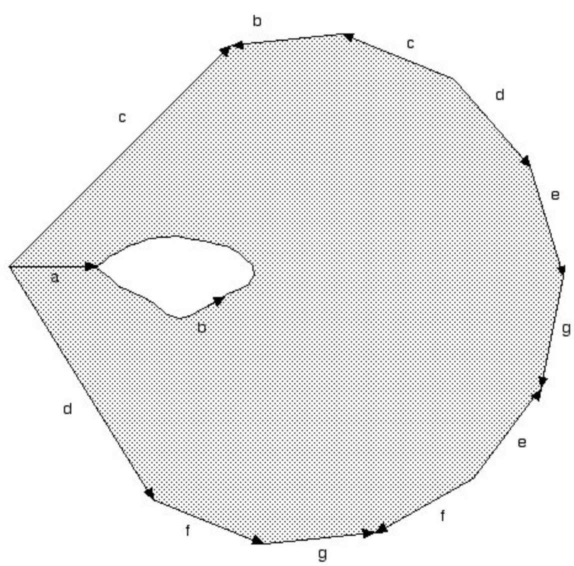


We now need to determine which vertices in the planar diagram become identified to the same point. We label the vertices, using the same label for those vertices which become identified, and we keep track of how many labels are needed. In this case we need four labels, P, Q, R and S since the 14 vertices get identified to 4 points. Based on how we broke the planar diagram into triangles, we see that there are 5 vertices (P, Q, R, S and the vertex at the center of the 14-gon), and there are $7+14$ edges (14 radiating from the center since it is a 14-gon, and 14/2 around the outside, again since it is a 14-gon), and 14 triangles (since it is a 14-gon), giving an Euler characteristic of $5-(7+14)+14 = -2 = 2-2g$, so the genus g is 2; i.e., it is a 2-holed torus. Note that we'd get the same answer for the Euler characteristic using the formula $v-n+1$ where the planar diagram is a $2n$-gon (in this case $n = 7$) and where v is the number of vertices in the planar diagram after identification (in this case 4), since $5-(7+14)+14 = (4+1)-(7+14)+14 = (v+1)-(n+14)+14 = v-n+1$. Thus you don't actually need to break the planar diagram up into triangles, you just need to determine v and n and use the formula $v-n+1$.

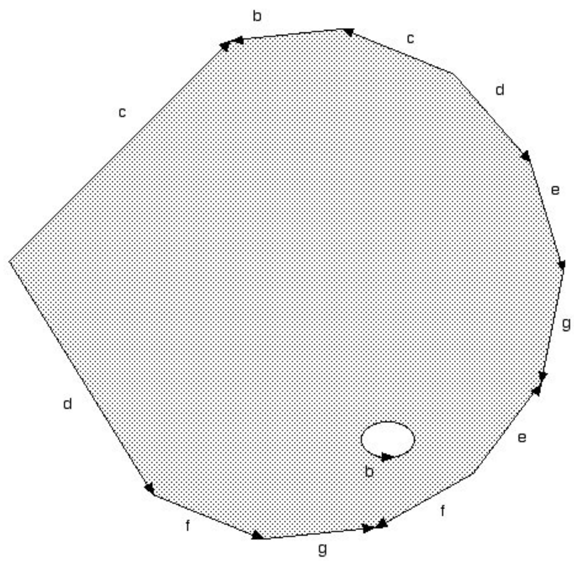
Here is an alternative approach, using an ad hoc sequence of steps that shows how to visualize what the planar diagram gives. Identify the heads of the two sides labeled a:



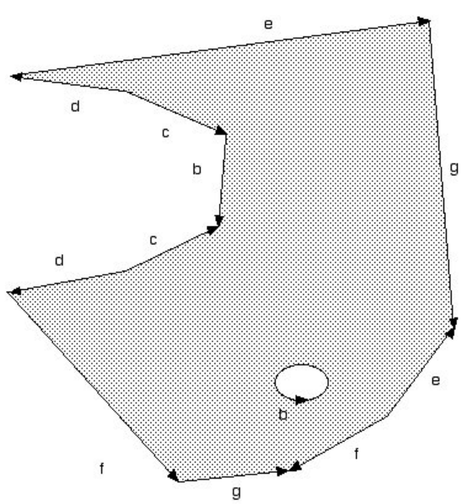
Now identify the two sides labeled a:



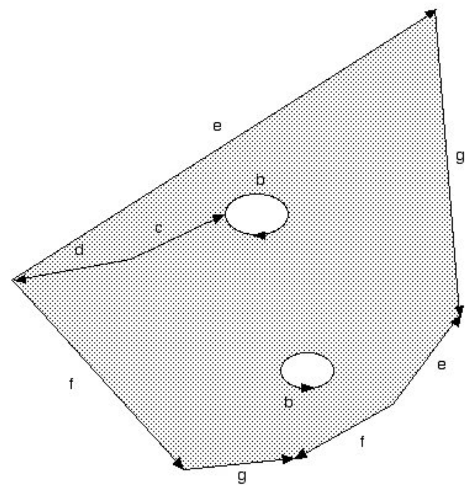
This makes a disappear, and we can slide the hole left by b out of the way:



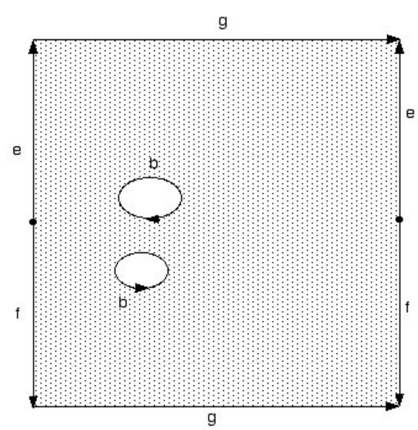
Now we bring the pairs of sides labeled c and d closer together in preparation for identifying them:



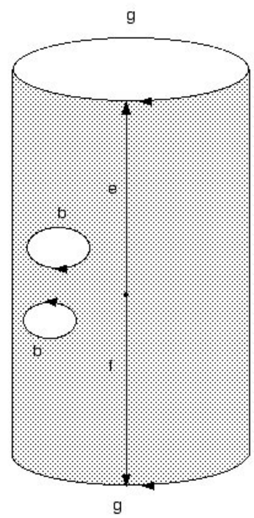
Now identify them:



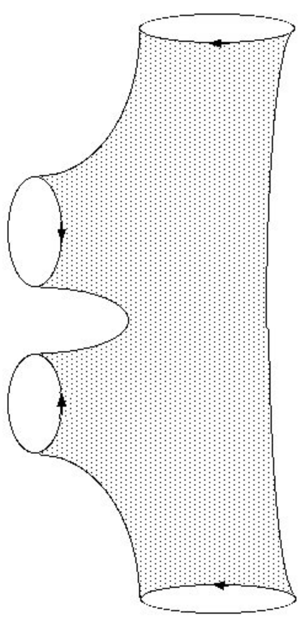
This makes sides c and d disappear:



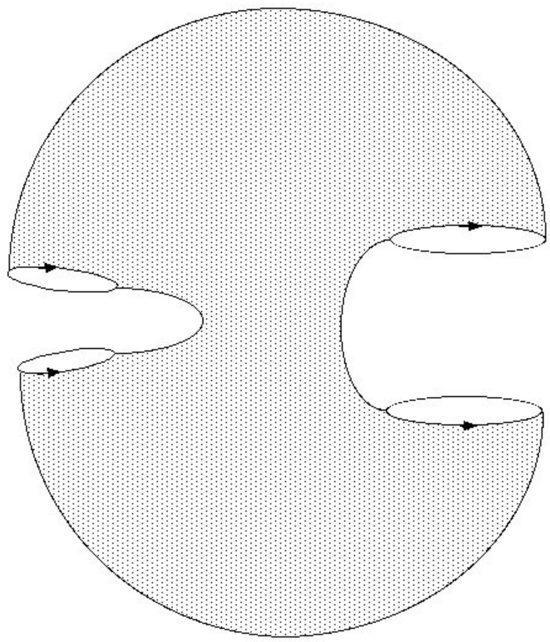
We next identify e and f:



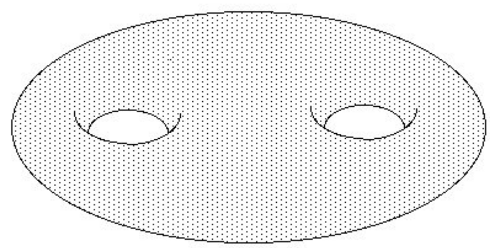
Pull the holes bounded by b outward:



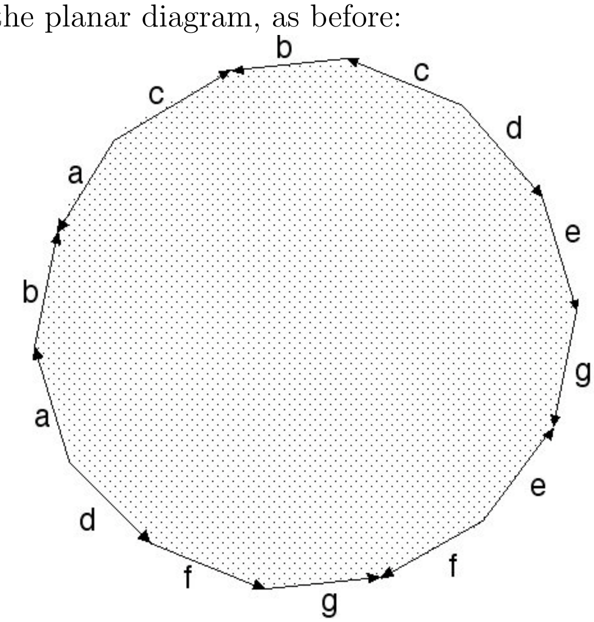
Bring the two b loops and the two g loops closer together:



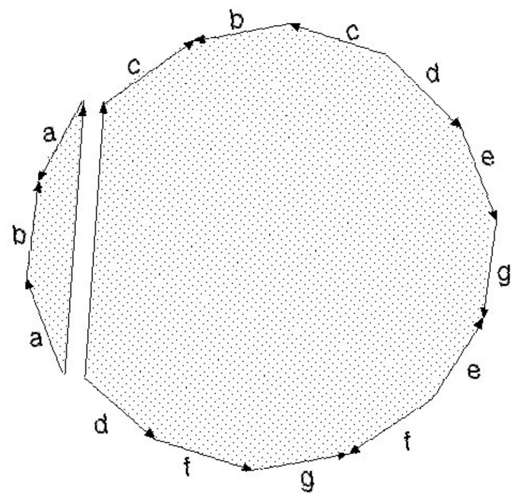
Identify the two b loops and the two g loops, and we're done, we have a two holed torus:



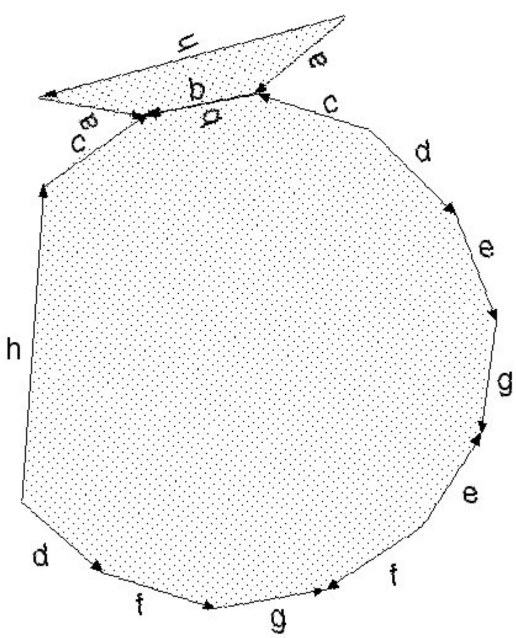
Here is an alternative approach using disassembly and reassembly. We start with the planar diagram, as before:



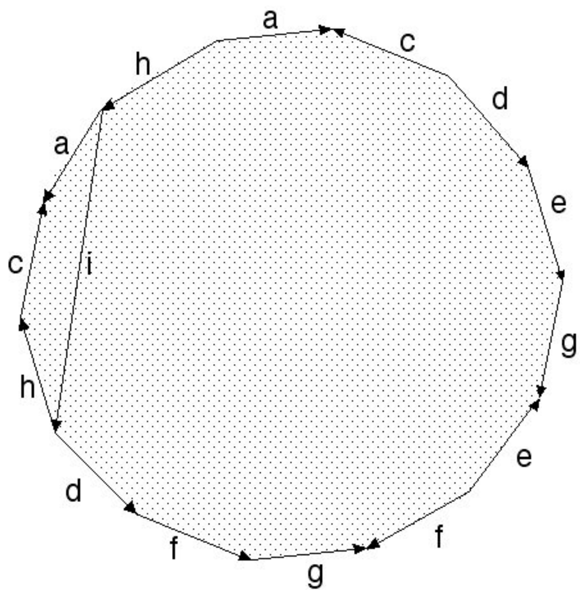
Cut off a piece that has both edges labeled a, by making a slice from the tail of one a edge to the tail of the other a (sometimes you may want the slice to go head to head, too). The slice creates two new edges, which we label h:



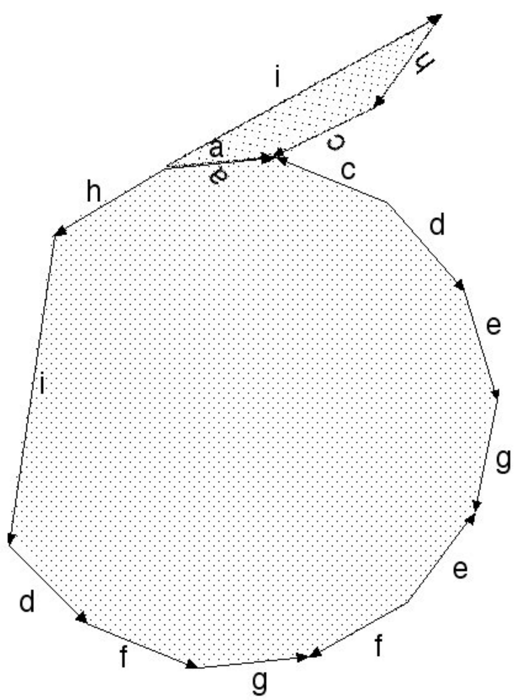
Reattach the piece, using an edge other than an a and not using the edge you created when you made the slice; in this case there's no choice, we have to reattach along b:



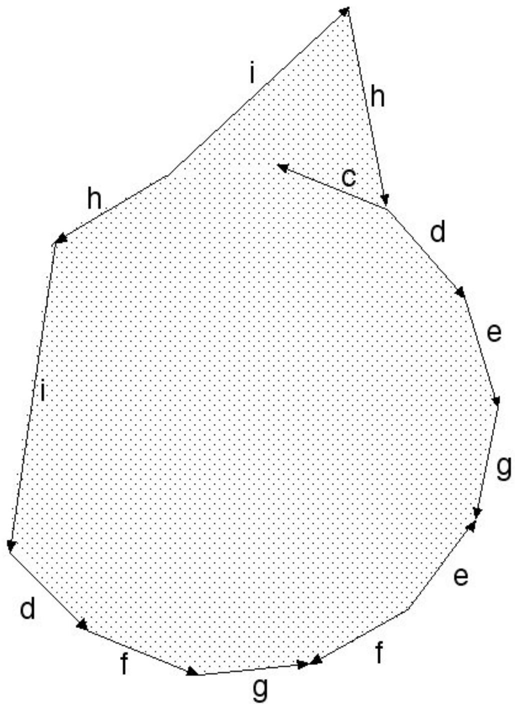
Now redraw it to make it easier to see what we're doing, and make a slice from head to foot on the newly created edges (labeled h):



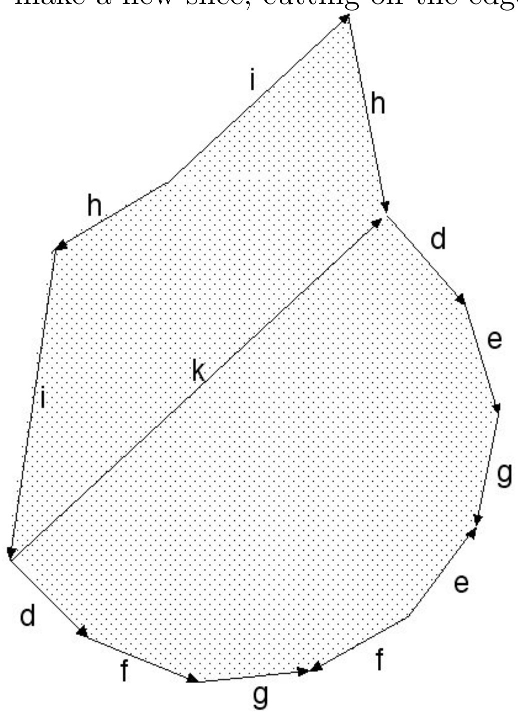
Reattach the sliced off piece by gluing the a pieces together:



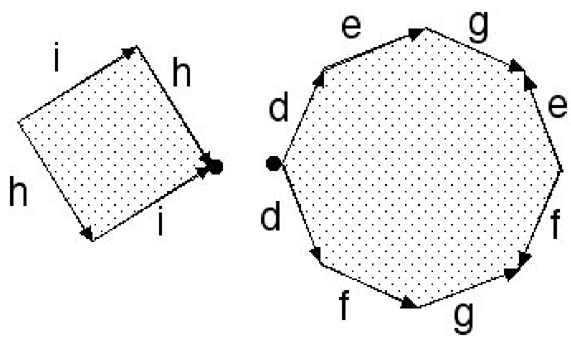
Since the c edges are adjacent and opposite we can glue them together (which will make them get absorbed into the interior, making them disappear):



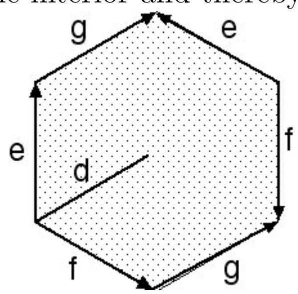
Now make a new slice, cutting off the edges labeled with i or h:



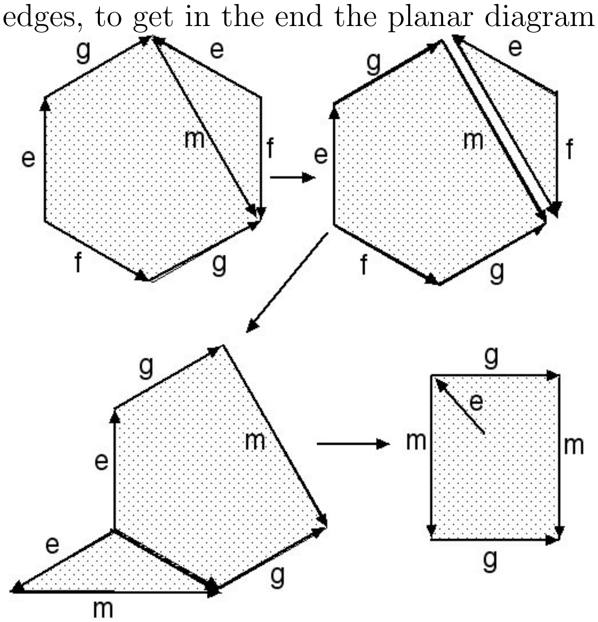
The two pieces we get each have an edge labeled k. By the reverse of the connected sum operation we can collapse the edges marked k to a point (shown as a dot) and our planar diagram splits into the connected sum of two planar diagrams, the left one being that for a one-holed torus, so we need to figure out what the right planar diagram gives:



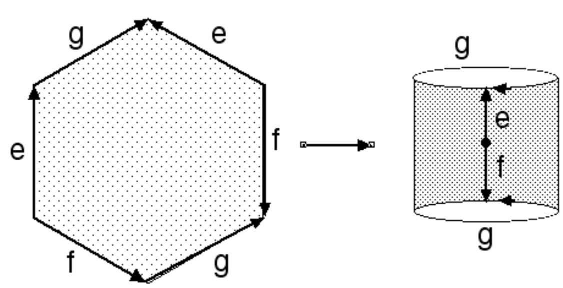
We can identify the two edges labeled d on the other piece, engulfing them into the interior and thereby making them disappear:



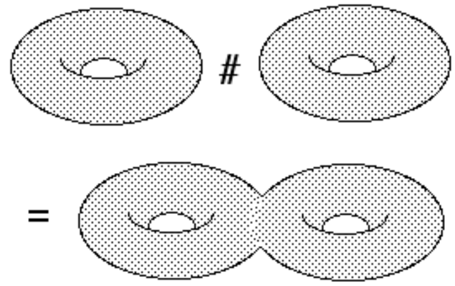
In the same way as before, we can make a head to head cut based on the g edges, and reattach the cut off piece, making the f edges disappear, then glue together the e edges, to get in the end the planar diagram for another one-holed torus:



Alternatively, we could have glued together the two edges labeled e and the two labeled f, giving the cylinder in the figure below.



Identifying the edges labeled g then gives us the one-holed torus again, so the planar diagram we started with gives the connected sum of two one-holed toruses (i.e., it gives a two-holed torus):



(6) Determine whether the points $(1,1,1), (2,3,4), (3,5,7) \in \mathbf{R}^3$ are geometrically independent.

Solution: The points are not geometrically independent, since $(2,3,4) - (1,1,1) = (1,2,3)$, $(3,5,7) - (1,1,1) = (2,4,6)$, but $(2,4,6) = 2(1,2,3)$. Alternatively, $2(1,2,3) - 1(2,4,6) = (0,0,0)$, so the vectors $(1,2,3), (2,4,6)$ are not linearly independent, hence the points $(1,1,1), (2,3,4), (3,5,7) \in \mathbf{R}^3$ are not geometrically independent.