Homework 6, due Thursday, October 25, 2012

Do any 5 of the 8 problems. Each problem is worth 20 points. Solutions will be graded for correctness, clarity and style.

(1) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d : X \times Y \to \mathbf{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

whenever $(x_1, y_1), (x_2, y_2) \in X \times Y$. Show that d is a metric.

Solution: First, $d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \geq d_X(x_1, x_2)$ but $d_X(x_1, x_2) \geq 0$ since d_X is a metric, so $d((x_1, y_1), (x_2, y_2)) \geq 0$ for all

$$(x_1, y_1), (x_2, y_2) \in X \times Y.$$

Second, if $(x_1, y_1) = (x_2, y_2)$, then $x_1 = x_2$ (so $d_X(x_1, x_2) = 0$) and $y_1 = y_2$ (so $d_Y(y_1, y_2) = 0$), hence $d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = 0$. Third, if $d((x_1, y_1), (x_2, y_2)) = 0$, then $\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = 0$, hence $d_X(x_1, x_2) = 0$ and $d_Y(y_1, y_2) = 0$, so $x_1 = x_2$ and $y_1 = y_2$, hence $(x_1, y_1) = (x_2, y_2)$. Fourth, for any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$ we have

$$d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3)$$

and

$$d_Y(y_1, y_3) \leq d_Y(y_1, y_2) + d_Y(y_2, y_3)$$

hence (a) $\max\{d_X(x_1, x_3), d_Y(y_1, y_3)\}$ is at most

$$\max\{d_X(x_1, x_2) + d_X(x_2, x_3), d_Y(y_1, y_2) + d_Y(y_2, y_3)\}$$

but $d_X(x_1, x_2) + d_X(x_2, x_3)$ is at most

$$\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} + \max\{d_X(x_2, x_3), d_Y(y_2, y_3)\}$$

and likewise $d_Y(y_1, y_2) + d_Y(y_2, y_3)$ is at most

$$\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} + \max\{d_X(x_2, x_3), d_Y(y_2, y_3)\}.$$

Thus (b) $\max\{d_X(x_1, x_2) + d_X(x_2, x_3), d_Y(y_1, y_2) + d_Y(y_2, y_3)\}$ is at most

$$\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} + \max\{d_X(x_2, x_3), d_Y(y_2, y_3)\}.$$

By putting (a) and (b) together we have that $\max\{d_X(x_1, x_3), d_Y(y_1, y_3)\}$ is at most

$$\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} + \max\{d_X(x_2, x_3), d_Y(y_2, y_3)\}$$

which is exactly $d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$.

(2) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d : X \times Y \to \mathbf{R}$ as in Problem 1, so $(X \times Y, d)$ is a metric space by Problem 1. Let $x \in X$ and $y \in Y$, and let $r > 0$ be a real number. Show that $D_{X\times Y}((x, y), r) = D_X(x, r) \times D_Y(y, r)$.

Solution: Let $(a, b) \in D_{X\times Y}((x, y), r)$. Then

$$d((x, y), (a, b)) = \max\{d_X(x, a), d_Y(y, b)\} < r,$$

hence $d_X(x, a) < r$ and $d_Y(y, b) < r$, so $a \in D_X(x, r)$ and $b \in D_Y(y, r)$, hence $(a, b) \in D_X(x, r) \times D_Y(y, r)$. Therefore $D_{X\times Y}((x, y), r) \subseteq D_X(x, r) \times D_Y(y, r)$

Now let $(a, b) \in D_X(x, r) \times D_Y(y, r)$. Then $a \in D_X(x, r)$ and $b \in D_Y(y, r)$, so $d_X(x, a) < r$ and $d_Y(y, b) < r$, hence $\max\{d_X(x, a), d_Y(y, b)\} < r$ and thus

$d((x,y),(a,b)) < r$ so $(a,b) \in D_{X\times Y}((x,y),r)$. Therefore $D_X(x,r) \times D_Y(y,r) \subseteq D_{X\times Y}((x,y),r)$.

This shows that $D_{X\times Y}((x,y),r) = D_X(x,r) \times D_Y(y,r)$.

(3) Let X and Y be sets. Let $U_i \subseteq X$ for all i in some set I and let $V_j \subseteq Y$ for all j in some set J. Let $U = \cup_{i\in I} U_i$ and let $V = \cup_{j\in J} V_j$. Show that $U \times V = \cup_{(i,j)\in I\times J} U_i \times V_j$.

Solution: Let $(a,b) \in U \times V = (\cup_{i\in I} U_i) \times (\cup_{j\in J} V_j)$. Thus $a \in \cup_{i\in I} U_i$ and hence $a \in U_{i'}$ for some $i' \in I$, and $b \in \cup_{j\in J} V_j$ and hence $b \in V_{j'}$ for some $j' \in J$. Thus $(a,b) \in U_{i'} \times V_{j'}$, hence $(a,b) \in \cup_{(i,j)\in I\times J} U_i \times V_j$. This shows $U \times V \subseteq \cup_{(i,j)\in I\times J} U_i \times V_j$.

Now let $(a,b) \in \cup_{(i,j)\in I\times J} U_i \times V_j$. Then for some $i' \in I$ and some $j' \in J$, we have $(a,b) \in U_{i'} \times V_{j'}$, so $a \in U_{i'}$ and $b \in V_{j'}$, hence $a \in U = \cup_{i\in I} U_i$ and $b \in V = \cup_{j\in J} V_j$, so $(a,b) \in U \times V$. This shows $\cup_{(i,j)\in I\times J} U_i \times V_j \subseteq U \times V$ and thus that $U \times V = \cup_{(i,j)\in I\times J} U_i \times V_j$.

(4) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d : X \times Y \to \mathbf{R}$ as in Problem 1, so $(X \times Y, d)$ is a metric space by Problem 1. Let $r > 0$ and let $s > 0$, and let $x \in X$ and let $y \in Y$. Show that $D_X(x,r) \times D_Y(y,s)$ is open in the metric topology on $X \times Y$.

Solution: It is enough for each $(a,b) \in D_X(x,r) \times D_Y(y,s)$ that there is a $t > 0$ sufficiently small so that $D_{X\times Y}((a,b),t) \subseteq D_X(x,r) \times D_Y(y,s)$. (From this it follows that $D_X(x,r) \times D_Y(y,s)$ is a union of open discs with respect to the metric d, and hence is open itself.) But $(a,b) \in D_X(x,r) \times D_Y(y,s)$ tells us that $a \in D_X(x,r)$ (and hence there is an $r' > 0$ sufficiently small so that $D_X(a,r') \subseteq D_X(x,r)$) and that $b \in D_Y(y,s)$ (and hence there is an $s' > 0$ sufficiently small so that $D_Y(b,s') \subseteq D_Y(y,s)$). Let $t = \min\{r', s'\}$. Then $D_{X\times Y}((a,b),t) = D_X(a,t) \times D_Y(b,t)$ by Problem 2, and $D_X(a,t) \times D_Y(b,t) \subseteq D_X(a,r') \times D_Y(b,s') \subseteq D_X(x,r) \times D_Y(y,s)$, as we wanted to show.

[Aside: How do we know that $a \in D_X(x,r)$ implies there is an $r' > 0$ sufficiently small so that $D_X(a,r') \subseteq D_X(x,r)$? Recall that this is by the triangle inequality. Let $p = d_X(x,a)$, and let $r' = r - p$. Then $c \in D_X(a,r')$ implies $d_X(a,c) < r'$, so $d_X(x,c) \leq d_X(x,a) + d_X(a,c) < p + r' = r$, hence $c \in D_X(x,r)$.]

(5) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d : X \times Y \to \mathbf{R}$ as in Problem 1, so $(X \times Y, d)$ is a metric space by Problem 1. Give X and Y their metric topologies, let $\mathcal{T}_{prod}$ be the product topology on $X \times Y$ and let $\mathcal{T}_{metric}$ be the metric topology on $X \times Y$ with respect to the metric d. Show that $\mathcal{T}_{prod} = \mathcal{T}_{metric}$. (I.e., if W is open in the metric topology on $X \times Y$, show that W is also open in the product topology, and vice versa. To do this think of W as a union of basis elements. Hint: Use Problem 2 to show that $W \in \mathcal{T}_{prod}$ if $W \in \mathcal{T}_{metric}$, and use Problems 3 and 4 to show that $W \in \mathcal{T}_{metric}$ if $W \in \mathcal{T}_{prod}$.)

Solution: Suppose W is open in the metric topology on $X \times Y$. Then W is a union of open sets of the form $D_{X\times Y}((x,y),r)$. But $D_{X\times Y}((x,y),r) = D_X(x,r) \times D_Y(y,r)$ and $D_X(x,r)$ by Problem 2, and $D_X(x,r)$ is open in X and $D_Y(y,r)$ is open in Y so $D_{X\times Y}((x,y),r) = D_X(x,r) \times D_Y(y,r)$ is open in the product topology on $X \times Y$. Since W is a union of sets open in the product topology, W itself is open in the product topology.

Now suppose that W is open in the product topology on $X \times Y$. Thus W is a union of sets of the form $U \times V$ where U is open in X and V is open in Y. But U, being open in X, is a union of sets of the form $D_X(x, r_x)$ and V, being open in Y, is a union of sets of the form $D_Y(y, r_y)$. Thus $U \times V$ is a union of sets of the form $D_X(x, r_x) \times D_Y(y, r_y)$ by Problem 3, and $D_X(x, r_x) \times D_Y(y, r_y)$ is open in the metric topology by Problem 4, so W also is open in the metric topology.

(6) Let $(\mathbf{R}, d_E)$ be the usual Euclidean metric on the real numbers (so $d_E(a, b) = |a - b|$) and let $d : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_E(x_1, x_2), d_E(y_1, y_2)\},$$

so d is a metric on $\mathbf{R}^2$ by Problem 1. Let δ_E be the Euclidean metric on $\mathbf{R}^2$, so

$$\delta_E((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Show that the metric topology on $\mathbf{R}^2$ with respect to d is the same as the metric topology on $\mathbf{R}^2$ with respect to δ_E. [Hint: Denote the open disc centered at $p \in \mathbf{R}^2$ of radius r with respect to d by $D_{(\mathbf{R}^2,d)}(p, r)$ and denote the open disc centered at $p \in \mathbf{R}^2$ of radius r with respect to δ_E by $D_{(\mathbf{R}^2,\delta_E)}(p, r)$. Show that $D_{(\mathbf{R}^2,d)}(p, r) \subseteq D_{(\mathbf{R}^2,\delta_E)}(p, r\sqrt{2})$ and that $D_{(\mathbf{R}^2,\delta_E)}(p, r) \subseteq D_{(\mathbf{R}^2,d)}(p, r)$.]

Solution: First we show that $D_{(\mathbf{R}^2,d)}(p, r) \subseteq D_{(\mathbf{R}^2,\delta_E)}(p, r\sqrt{2})$. Let $q \in D_{(\mathbf{R}^2,d)}(p, r)$. Say $p = (a_1, b_1)$ and $q = (a_2, b_2)$. Then $|a_i - b_i| \le d(p.q) < r$, so $\delta_E(p, q) = \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} < \sqrt{2r^2} = r\sqrt{2}$, hence $q \in D_{(\mathbf{R}^2,\delta_E)}(p, r\sqrt{2})$.

Now we show that $D_{(\mathbf{R}^2,\delta_E)}(p, r) \subseteq D_{(\mathbf{R}^2,d)}(p, r)$. Let $q \in D_{(\mathbf{R}^2,\delta_E)}(p, r)$. Thus $|a_i - b_i| \le \sqrt{|a_i - b_i|^2} \le \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2}$ for $i = 1, 2$, so $d(p, q) \le \delta_E(p, q) < r$, hence $q \in D_{(\mathbf{R}^2,d)}(p, r)$.

Now let W be open with respect to the metric d. Then W is a union of open discs of the form $D_{(\mathbf{R}^2,d)}(p, s)$ with $s > 0$. But for each $q \in D_{(\mathbf{R}^2,d)}(p, s)$, there is an $r > 0$ sufficiently small so that $D_{(\mathbf{R}^2,d)}(q, r) \subseteq D_{(\mathbf{R}^2,d)}(p, s)$. However, $D_{(\mathbf{R}^2,\delta_E)}(q, r) \subseteq D_{(\mathbf{R}^2,d)}(q, r)$, so $D_{(\mathbf{R}^2,d)}(p, s)$ (and hence W) is a union of open discs of the form $D_{(\mathbf{R}^2,\delta_E)}(q, r)$, hence W is open with respect to the metric δ_E.

Finally, let W be open with respect to the metric δ_E. Then W is a union of open discs of the form $D_{(\mathbf{R}^2,\delta_E)}(p, s)$ with $s > 0$. But for each $q \in D_{(\mathbf{R}^2,\delta_E)}(p, s)$, there is an $r > 0$ sufficiently small so that $D_{(\mathbf{R}^2,\delta_E)}(q, r) \subseteq D_{(\mathbf{R}^2,\delta_E)}(p, s)$. However, $D_{(\mathbf{R}^2,d)}(q, r/\sqrt{2}) \subseteq D_{(\mathbf{R}^2,\delta_E)}(q, r)$, so $D_{(\mathbf{R}^2,\delta_E)}(p, s)$ (and hence W) is a union of open discs of the form $D_{(\mathbf{R}^2,d)}(q, r)$, hence W is open with respect to the metric d.

(7) Let X, Y, Z and W be topological spaces. Let $f : X \to Z$ and $g : Y \to W$ be continuous maps. Define $H : X \times Y \to Z \times W$ by $H((x, y)) = (f(x), g(y))$. Show that H is continuous if we give $X \times Y$ and $Z \times W$ the product topologies.

Solution: If $V \subseteq Z \times W$, we must show that $H^{-1}(V)$ is open. But V is a union of open sets of the form $V_1 \times V_2$, where $V_1 \subseteq Z$ is open and $V_2 \subseteq W$ is open, so it is enough to show that $H^{-1}(V_1 \times V_2)$ is open. Now, $(x, y) \in H^{-1}(V_1 \times V_2)$ if and only if $(f(x), g(y)) = H((x, y)) \in V_1 \times V_2$ if and only if $f(x) \in V_1$ and $g(y) \in V_2$ if and only if $x \in f^{-1}(V_1)$ and $y \in g^{-1}(V_2)$ if and only if $(x, y) \in f^{-1}(V_1) \times g^{-1}(V_2)$. I.e., $H^{-1}(V_1 \times V_2) = f^{-1}(V_1) \times g^{-1}(V_2)$, but $f^{-1}(V_1)$ is open in X since f is continuous and $g^{-1}(V_2)$ is open in Y since g is continuous, so $f^{-1}(V_1) \times g^{-1}(V_2)$ is open in $X \times Y$.

(8) Let X and Y be topological spaces and give $\mathbf{R}$ and $\mathbf{R}^2$ the standard topologies. Let $f : X \to \mathbf{R}$ and $g : Y \to \mathbf{R}$ be continuous maps. You may assume that $p : \mathbf{R}^2 \to \mathbf{R}$, defined by $p((a, b)) = a + b$, is continuous. Define $h : X \times Y \to \mathbf{R}$ by $h((x, y)) = f(x) + g(y)$. Show that h is continuous. [Hint: Use the fact that the standard topology on $\mathbf{R}^2$ is the product topology to apply Problem 7, using the fact that $h = p \circ H$, where $H : X \times Y \to \mathbf{R}^2$ is defined by $H((x, y)) = (f(x), g(y))$.]

Solution: Since the standard topology on $\mathbf{R}^2$ is the product topology, H is continuous by Problem 7, and since p is continuous and compositions of continuous functions are continuous, $h = p \circ H$ is continuous.