Homework 6, due Thursday, October 25, 2012

Do any 5 of the 8 problems. Each problem is worth 20 points. Solutions will be graded for correctness, clarity and style.

(1) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d : X \times Y \to \mathbf{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

whenever $(x_1, y_1), (x_2, y_2) \in X \times Y$. Show that d is a metric.

(2) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d : X \times Y \to \mathbf{R}$ as in Problem 1, so $(X \times Y, d)$ is a metric space by Problem 1. Let $x \in X$ and $y \in Y$, and let $r > 0$ be a real number. Show that $D_{X\times Y}((x, y), r) = D_X(x, r) \times D_Y(y, r)$.

(3) Let X and Y be sets. Let $U_i \subseteq X$ for all i in some set I and let $V_j \subseteq Y$ for all j in some set J. Let $U = \cup_{i\in I} U_i$ and let $V = \cup_{j\in J} V_j$. Show that $U \times V = \cup_{(i,j)\in I\times J} U_i \times V_j$.

(4) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d : X \times Y \to \mathbf{R}$ as in Problem 1, so $(X \times Y, d)$ is a metric space by Problem 1. Let $r > 0$ and let $s > 0$, and let $x \in X$ and let $y \in Y$. Show that $D_X(x, r) \times D_Y(y, s)$ is open in the metric topology on $X \times Y$.

(5) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d : X \times Y \to \mathbf{R}$ as in Problem 1, so $(X \times Y, d)$ is a metric space by Problem 1. Give X and Y their metric topologies, let $\mathcal{T}_{prod}$ be the product topology on $X \times Y$ and let $\mathcal{T}_{metric}$ be the metric topology on $X \times Y$ with respect to the metric d. Show that $\mathcal{T}_{prod} = \mathcal{T}_{metric}$. (I.e., if W is open in the metric topology on $X \times Y$, show that W is also open in the product topology, and vice versa. To do this think of W as a union of basis elements. Hint: Use Problem 2 to show that $W \in \mathcal{T}_{prod}$ if $W \in \mathcal{T}_{metric}$, and use Problems 3 and 4 to show that $W \in \mathcal{T}_{metric}$ if $W \in \mathcal{T}_{prod}$.)

(6) Let $(\mathbf{R}, d_E)$ be the usual Euclidean metric on the real numbers (so $d_E(a, b) = |a - b|$) and let $d : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_E(x_1, x_2), d_E(y_1, y_2)\},$$

so d is a metric on $\mathbf{R}^2$ by Problem 1. Let δ_E be the Euclidean metric on $\mathbf{R}^2$, so

$$\delta_E((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Show that the metric topology on $\mathbf{R}^2$ with respect to d is the same as the metric topology on $\mathbf{R}^2$ with respect to δ_E. [Hint: Denote the open disc centered at $p \in \mathbf{R}^2$ of radius r with respect to d by $D_{(\mathbf{R}^2,d)}(p, r)$ and denote the open disc centered at $p \in \mathbf{R}^2$ of radius r with respect to δ_E by $D_{(\mathbf{R}^2,\delta_E)}(p, r)$. Show that $D_{(\mathbf{R}^2,d)}(p, r) \subseteq D_{(\mathbf{R}^2,\delta_E)}(p, r\sqrt{2})$ and that $D_{(\mathbf{R}^2,\delta_E)}(p, r) \subseteq D_{(\mathbf{R}^2,d)}(p, r)$.]

(7) Let X, Y, Z and W be topological spaces. Let $f : X \to Z$ and $g : Y \to W$ be continuous maps. Define $H : X \times Y \to Z \times W$ by $H((x, y)) = (f(x), g(y))$. Show that H is continuous.

(8) Let X and Y be topological spaces and give $\mathbf{R}$ and $\mathbf{R}^2$ the standard topologies. Let $f : X \to \mathbf{R}$ and $g : Y \to \mathbf{R}$ be continuous maps. You may assume that $p : \mathbf{R}^2 \to \mathbf{R}$, defined by $p((a,b)) = a + b$, is continuous. Define $h : X \times Y \to \mathbf{R}$ by $h((x,y)) = f(x) + g(y)$. Show that h is continuous. [Hint: Use the fact that the standard topology on $\mathbf{R}^2$ is the product topology to apply Problem 7, using the fact that $h = p \circ H$, where $H : X \times Y \to \mathbf{R}^2$ is defined by $H((x,y)) = (f(x), g(y))$.]