Homework 5, due Thursday, October 11, 2012

Do any 5 of the 8 problems. Each problem is worth 20 points. Solutions will be graded for correctness, clarity and style.

(1) Let X be a topological space. If C is a finite subset of X, show that C is compact.

Solution: Let $C = \{c_1, \dots, c_r\}$. Given any open cover $\{U_i : i \in I\}$ of C, for each $1 \le j \le r$, pick an open set U_{i_j} such that $c_j \in U_{i_j}$. Then $\{U_{i_1}, \dots, U_{i_r}\}$ is a finite open cover of C. Thus any open of C has a finite subcover so C is compact.

(2) Let X be a set with the discrete topology. If $C \subseteq X$ is compact, show that C is finite.

Solution: Since $\{x\}$ is open for each $x \in X$, we have an open cover $\{\{c\} : c \in C\}$. Since C is compact, $\{\{c\} : c \in C\}$ must have a finite subcover. But no proper subset of $\{\{c\} : c \in C\}$ covers C, so the only way for $\{\{c\} : c \in C\}$ to have a finite subcover is for $\{\{c\} : c \in C\}$ itself to be finite, which means that C is finite.

(3) Let X be a topological space. Show that X is a T_1-space if and only if each point of X is a closed set.

Solution: Assume X is a T_1-space. Let $x \in X$. We will show that $\{x\}^c = X \backslash \{x\}$ is open. Let $y \in \{x\}^c$. Since X is a T_1-space, we know there is an open neighborhood U_y of y that does not contain x. Thus $U_y \subseteq \{x\}^c$, so $\{x\}^c$ contains an open neighborhood of each of its points. Thus $\{x\}^c$ is open so $\{x\}$ is closed.

Conversely, assume each point of X is closed. Let $x, y \in X$, $x \neq y$. Since $\{x\}$ is closed, $\{x\}^c$ is open and since $x \neq y$, we have $y \in \{x\}^c$. Similarly, since $\{y\}$ is closed, $\{y\}^c$ is open and since $y \neq x$, we have $x \in \{y\}^c$. Thus each point is contained in an open set that does not contain the other, so X is a T_1-space.

(4) Give a direct proof that a metric space (X, d) is Hausdorff. (Do not for example use the fact that a metric space is a T_3-space and every T_3-space is a T_2-space.)

Solution: Let $x, y \in X$, $x \neq y$. Let $r = d(x, y)/2$. Then $D_X(x, r)$ is an open neighborhood of x, and $D_X(y, r)$ is an open neighborhood of y, but $D_X(x, r) \cap D_X(x, r) = \varnothing$. (This is because if $z \in D_X(x, r) \cap D_X(x, r)$, then $d(x, z) < r$ and $d(y, z) < r$, so $d(x, z) + d(z, y) < 2r = d(x, y)$, but this contradicts the triangle inequality.)

(5) Let $f : X \to Y$ be a continuous bijective map of topological spaces. Note that since f is bijective we can define the inverse function $f^{-1} : Y \to X$ as $f^{-1}(y) = x$ whenever $f(x) = y$. If X is compact and Y is Hausdorff, show that f^{-1} is also continuous. [Aside: when a continuous bijective map $f : X \to Y$ has a continuous inverse, we say that the map f is a *homeomorphism* and that X and Y are *homeomorphic*.]

Solution: It is enough to show that $(f^{-1})^{-1}(C)$ is closed for any closed subset $C \subseteq X$. But $(f^{-1})^{-1} = f$, so $(f^{-1})^{-1}(C) = f(C)$. Since X is compact and C is closed, C is compact, so $f(C)$ is compact, and since Y is Hausdorff, any compact subset of Y is closed. In particular, $f(C)$ is closed. Thus f^{-1} is continuous.

(6) Give an example of a continuous bijective map $f : X \to Y$ where X is compact but Y is not Hausdorff and where f^{-1} is not continuous.

Solution: Let X be a finite set of at least 2 points with the discrete topology, and let Y be a finite set with $|Y| = |X|$ but give Y the indiscrete topology. Let $f : X \to Y$ be any bijection. Then f is continuous since $f^{-1}(Y) = X$ and $f^{-1}(\varnothing) = \varnothing$, so the inverse image of any open set is open. Now let $x \in X$. Thus $\{x\}$ is open, but $(f^{-1})^{-1}(\{x\}) = f(\{x\})$ is not open, since $f(\{x\})$ is not $\varnothing$ and not Y, hence not open in Y. Thus f^{-1} is not continuous.

(7) Let $X = \mathbf{R}$ have the standard topology, and let $Y = (0, \infty) \subset X$ have the subspace topology. Show that X and Y are homeomorphic. (I.e., find a continuous bijective map $f : X \to Y$ with a continuous inverse. You do not need to prove that your f is continuous, bijective or has a continuous inverse, but it should be obvious to a Calc I student that the f that you pick is continuous, bijective and has a continuous inverse.)

Solution: Let $f : X \to Y$ be $f(x) = e^x$. This is continuous and bijective and has a continuous inverse $f^{-1}(y) = \log_e(y)$.

(8) Let (X, d) be a metric space and let $a \in X$. Define $f : X \to \mathbf{R}$ by $f(x) = d(a, x)$ for all $x \in X$. If $\mathbf{R}$ has the standard topology, show that f is continuous.

Solution: Let $V \subseteq \mathbf{R}$ be open. We will show that $f^{-1}(V)$ is open. It suffices to show that $f^{-1}(V)$ contains an open neighborhood of each of its points. So let $b \in f^{-1}(V)$. Since $f(b) \in V$, there is an $\epsilon > 0$ such that the open interval

$$(f(b) - \epsilon, f(b) + \epsilon)$$

is contained in V.

Let $x \in D_X(b, \epsilon)$. Then $f(x) = d(a, x) \leq d(a, b) + d(b, x) < f(b) + \epsilon$, and $f(b) = d(a, b) \leq d(a, x) + d(x, b) < f(x) + \epsilon$ which gives $f(x) > f(b) - \epsilon$. Thus $f(x) \in (f(b) - \epsilon, f(b) + \epsilon) \subseteq V$, so $f(D_X(b, \epsilon)) \in V$, hence $D_X(b, \epsilon) \in f^{-1}(V)$. I.e., $D_X(b, \epsilon)$ is a neighborhood of b contained in $f^{-1}(V)$, so $f^{-1}(V)$ contains an open neighborhood of each of its points and hence is open.