Homework 5, due Thursday, October 11, 2012

Do any 5 of the 8 problems. Each problem is worth 20 points. Solutions will be graded for correctness, clarity and style.

(1) Let X be a topological space. If C is a finite subset of X, show that C is compact.

(2) Let X be a set with the discrete topology. If $C \subseteq X$ is compact, show that C is finite.

(3) Let X be a topological sapce. Show that X is a T_1-space if and only if each point of X is a closed set.

(4) Give a direct proof that a metric space (X, d) is Hausdorff. (Do not for example use the fact that a metric space is a T_3-space and every T_3-space is a T_2-space.)

(5) Let $f : X \to Y$ be a continuous bijective map of topological spaces. Note that since f is bijective we can define the inverse function $f^{-1} : Y \to X$ as $f^{-1}(y) = x$ whenever $f(x) = y$. If X is compact and Y is Hausdorff, show that f^{-1} is also continuous. [Aside: when a continuous bijective map $f : X \to Y$ has a continuous inverse, we say that the map f is a *homeomorphism* and that X and Y are *homeomorphic*.]
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(6) Give an example of a continuous bijective map $f : X \to Y$ where X is compact but Y is not Hausdorff and where f^{-1} is not continuous.

(7) Let $X = \mathbf{R}$ have the standard topology, and let $Y = (0, \infty) \subset X$ have the subspace topology. Show that X and Y are homeomorphic. (I.e., find a continuous bijective map $f : X \to Y$ with a continuous inverse. You do not need to prove that your f is continuous, bijective or has a continuous inverse, but it should be obvious to a Calc I student that the f that you pick is continuous, bijective and has a continuous inverse.)

(8) Let (X, d) be a metric space and let $a \in X$. Define $f : X \to \mathbf{R}$ by $f(x) = d(a, x)$ for all $x \in X$. If $\mathbf{R}$ has the standard topology, show that f is continuous.