Homework 4, due Thursday, September 20, 2012

Given sets X and Y, define the product $X \times Y$ to be the set $X \times Y = \{(x, y) : x \in X, y \in Y\}$, and define the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ by $\pi_X((x, y)) = x$ and $\pi_Y((x, y)) = y$ for all $(x, y) \in X \times Y$.

Now assume $\mathcal{T}_X$ is a topology on X and $\mathcal{T}_Y$ is a topology on Y. Let $\mathcal{B}_{X \times Y}$ be the collection of all subsets of $X \times Y$ of the form $U \times V$, where $U \subseteq X$ is open in X (i.e., $U \in \mathcal{T}_X$) and $V \subseteq Y$ is open in Y (i.e., $V \in \mathcal{T}_Y$). Let $\mathcal{T}_{X \times Y}$ be the collection of all unions of elements of $\mathcal{B}_{X \times Y}$. Then $\mathcal{T}_{X \times Y}$ is a topology on $X \times Y$ called the product topology, and $\mathcal{B}_{X \times Y}$ is a basis for $\mathcal{T}_{X \times Y}$

Do any 4 of the 6 problems. Each problem is worth 25 points. Solutions will be graded for correctness, clarity and style.

(1) Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined as

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Intuitively, we know that f is not continuous. Prove it by finding a closed subset $B \subseteq \mathbf{R}$ such that $f^{-1}(B)$ is not closed and by finding an open subset $V \subseteq \mathbf{R}$ such that $f^{-1}(V)$ is not open

Solution: Let $B = [1, 4]$. Then $f^{-1}(B) = [-2, -1] \cup (1, 2]$ is not closed since $1 \in \mathrm{Cl}(f^{-1}(B))$ but $1 \notin f^{-1}(B)$. And let $V = B^c$; then V is open since B is closed. Then $f^{-1}(V) = f^{-1}(B^c) = (f^{-1}(B))^c = (-\infty, -2) \cup (-1, 1] \cup (2, \infty)$ is not open, since $(f^{-1}(V))^c = ((f^{-1}(B))^c)^c = f^{-1}(B)$ is not closed.

(2) Let X and Y be sets, and let $A \subseteq X$ and $B \subseteq Y$ be subsets. Prove that $(\pi_X)^{-1}(A) = A \times Y$ and $(\pi_Y)^{-1}(B) = X \times B$.

Solution: By definition, $(\pi_X)^{-1}(A) = \{(x, y) \in X \times Y : \pi_X((x, y)) \in A\}$, but $\{(x, y) \in X \times Y : \pi_X((x, y)) \in A\} = \{(x, y) \in X \times Y : x \in A\} = A \times Y$. Similarly, $(\pi_Y)^{-1}(B) = \{(x, y) \in X \times Y : \pi_Y((x, y)) \in B\}$, but $\{(x, y) \in X \times Y : \pi_Y((x, y)) \in B\} = \{(x, y) \in X \times Y : y \in B\} = X \times B$.

(3) Let X and Y be topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively. Let $X \times Y$ have the product topology. Prove that π_X and π_Y are continuous.

Solution: Let $U \subseteq X$ be open. Then $(\pi_X)^{-1}(U) = U \times Y$ is in $\mathcal{B}_{X \times Y} \subseteq \mathcal{T}_{X \times Y}$, hence is open. Thus π_X is continuous. Similarly, let $V \subseteq Y$ be open. Then $(\pi_Y)^{-1}(V) = X \times V$ is in $\mathcal{B}_{X \times Y} \subseteq \mathcal{T}_{X \times Y}$, hence is open. Thus π_Y is continuous.

(4) Let $f : X \to Y$ be a map of topological spaces. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Prove that f is continuous if and only if $f^{-1}(V)$ is open in X for every $V \in \mathcal{B}_Y$.

Solution: If f is continuous, then $f^{-1}(V)$ is open for every open subset V of Y. But every element of $\mathcal{B}_Y$ is an open subset of Y, so $f^{-1}(V)$ is open for every $V \in \mathcal{B}_Y$.

Conversely, assume that $f^{-1}(V)$ is open in X for every $V \in \mathcal{B}_Y$. We must show that $f^{-1}(W)$ is open for every open subset W of Y. But every open subset is a union of basis elements, so given an open subset $W \subseteq Y$, we have $W = \cup_{i \in I} V_i$ for some collection V_i, $i \in I$ of basis elements. Now $f^{-1}(W) = f^{-1}(\cup_{i \in I} V_i) = \cup_{i \in I} f^{-1}(V_i)$, so W is a union of open subsets of X hence open in X itself.

(5) For any sets A, B and C, prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ (i.e., intersection distributes over union).

Solution: Let $x \in (A \cup B) \cap C$. Then $x \in C$ and $x \in A \cup B$, so either $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cap C \subseteq (A \cap C) \cup (B \cap C)$ while if $x \in B$, then $x \in B \cap C \subseteq (A \cap C) \cup (B \cap C)$.

Now assume $x \in (A \cap C) \cup (B \cap C)$. Then either $x \in A \cap C$ or $x \in B \cap C$. If $x \in A \cap C$, then $x \in A \subseteq A \cup B$ and $x \in C$, so $x \in (A \cup B) \cap C$. If $x \in B \cap C$, then $x \in B \subseteq A \cup B$ and $x \in C$, so $x \in (A \cup B) \cap C$.

(6) For any sets A, B and C, prove that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ (i.e., union distributes over intersection).

Solution: Let $x \in (A \cap B) \cup C$. Then $x \in C$ or $x \in A \cap B$. If $x \in C$, then $x \in A \cup C$ and $x \in B \cup C$, so $x \in (A \cup C) \cap (B \cup C)$.

Now assume $x \in (A \cup C) \cap (B \cup C)$. Then $x \in A \cup C$ and $x \in B \cup C$. If $x \in C$, then $x \in (A \cap B) \cup C$. If x is not in C, then $x \in A$ (since $x \in A \cup C$) and $x \in B$ (since $x \in B \cup C$). Thus $x \in A \cap B \subseteq (A \cap B) \cup C$.