Homework 4, due Thursday, September 20, 2012

Given sets X and Y, define the product $X \times Y$ to be the set $X \times Y = \{(x, y) : x \in X, y \in Y\}$, and define the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ by $\pi_X((x, y)) = x$ and $\pi_Y((x, y)) = y$ for all $(x, y) \in X \times Y$.

Now assume $\mathcal{T}_X$ is a topology on X and $\mathcal{T}_Y$ is a topology on Y. Let $\mathcal{B}_{X\times Y}$ be the collection of all subsets of $X \times Y$ of the form $U \times V$, where $U \subseteq X$ is open in X (i.e., $U \in \mathcal{T}_X$) and $V \subseteq Y$ is open in Y (i.e., $V \in \mathcal{T}_Y$). Let $\mathcal{T}_{X\times Y}$ be the collection of all unions of elements of $\mathcal{B}_{X\times Y}$. Then $\mathcal{T}_{X\times Y}$ is a topology on $X \times Y$ called the product topology, and $\mathcal{B}_{X\times Y}$ is a basis for $\mathcal{T}_{X\times Y}$

Do any 4 of the 6 problems. Each problem is worth 25 points. Solutions will be graded for correctness, clarity and style.

(1) Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined as
$$f(x) = \begin{cases} x^2 & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}$$
Intuitively, we know that f is not continuous. Prove it by finding a closed subset $B \subseteq \mathbf{R}$ such that $f^{-1}(B)$ is not closed and by finding an open subset $V \subseteq \mathbf{R}$ such that $f^{-1}(V)$ is not open

(2) Let X and Y be sets, and let $A \subseteq X$ and $B \subseteq Y$ be subsets. Prove that $(\pi_X)^{-1}(A) = A \times Y$ and $(\pi_Y)^{-1}(B) = X \times B$.

(3) Let X and Y be topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively. Let $X \times Y$ have the product topology. Prove that π_X and π_Y are continuous.

(4) Let $f : X \to Y$ be a map of topological spaces. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Prove that f is continuous if and only if $f^{-1}(V)$ is open in X for every $V \in \mathcal{B}_Y$.

(5) For any sets A, B and C, prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ (i.e., intersection distributes over union).

(6) For any sets A, B and C, prove that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ (i.e., union distributes over intersection).