Homework 3, due Thursday, September 13, 2012

Let X be a metric space with metric d. Let $S \subseteq X$. We say that S is *bounded* if for some $r > 0$ and $x \in X$ we have $S \subseteq D_X(x, r)$.

If X is a topological space and $S \subseteq X$, note that $\mathrm{Cl}(S) = S$ if S is closed. I.e., the closure of a closed set is the set itself. [We had two equivalent definitions of the closure $\mathrm{Cl}(S)$ of a set S. You can use whichever one you prefer. The first was that $\mathrm{Cl}(S)$ consists of all $x \in X$ such that every open neighborhood of x meets S. The second was that $\mathrm{Cl}(S)$ is the intersection of all closed subsets of X that contain S. Here is a proof that $\mathrm{Cl}(S) = S$ if S is closed, using definition 1: if $x \in S$ then every open neighborhood U of x clearly meets S (since x is in both U and S), and hence (by definition 1) $x \in \mathrm{Cl}(S)$. Thus $S \subseteq \mathrm{Cl}(S)$. And if $x \in \mathrm{Cl}(S)$, then every open neighborhood of x meets S (by definition 1). If $x \notin S$, then $x \in S^c$ and S^c is open (since S is closed), so S^c is an open neighborhood of x that does not meet S, which is a contradiction. Thus we must have $x \in S$, and hence $\mathrm{Cl}(S) \subseteq S$. Now here is a proof that $\mathrm{Cl}(S) = S$ if S is closed, using definition 2: by definition 2, $\mathrm{Cl}(S) = \cap_{C \in A} C$, where A is the set of all closed subsets of X that contain S, so $S \subseteq \mathrm{Cl}(S)$ since $\mathrm{Cl}(S)$ is an intersection of sets C each of which contains S. And $\mathrm{Cl}(S) \subseteq S$ since one of the sets C whose intersection is $\mathrm{Cl}(S)$ is S itself, since by hypothesis S is a closed subset of X which contains S.]

Do any 5 of the 6 problems. Each problem is worth 20 points. Solutions will be graded for correctness, clarity and style.

Problem 1: Let X be a topological space and let $S \subseteq X$. Show that $\mathrm{Fr}(S)$ is closed. (Recall that $\mathrm{Fr}(S)$ consists of every $x \in X$ such that every open neighborhood of x meets both S and S^c.)

Solution: We must show $(\mathrm{Fr}(S))^c$ is open. Let $x \in (\mathrm{Fr}(S))^c$. By definition this means that some open neighborhood U_x of x either does not meet S or does not meet S^c. Since U_x is an open neighborhood of each of its points, every point $u \in U_x$ is in $(\mathrm{Fr}(S))^c$ and hence $U_x \subseteq (\mathrm{Fr}(S))^c$. Therefore $(\mathrm{Fr}(S))^c$ is open since it contains an open neighborhood of each of its points (in fact, $(\mathrm{Fr}(S))^c = \cup_{x \in (\mathrm{Fr}(S))^c} U_x$ so $(\mathrm{Fr}(S))^c$ is a union of open subsets).

Here's an alternative solution. Note that $\mathrm{Cl}(S)$ consists of all $x \in X$ such that every neighborhood of x meets S, and $\mathrm{Cl}(S^c)$ consists of all $x \in X$ such that every neighborhood of x meets S^c, Thus $\mathrm{Cl}(S) \cap \mathrm{Cl}(S)$ consists of all $x \in X$ such that every neighborhood of x meets both S and S^c, which is the same as $\mathrm{Fr}(S)$. Thus $\mathrm{Fr}(S) = \mathrm{Cl}(S) \cap \mathrm{Cl}(S)$. But $\mathrm{Cl}(S)$ and $\mathrm{Cl}(S^c)$ are closed, and the intersection of closed sets is closed, so $\mathrm{Fr}(C)$ is closed.

Problem 2: Let X be a topological space and let $S \subseteq X$. Show that $\mathrm{Cl}(S) = \mathrm{Int}(S) \cup \mathrm{Fr}(S)$. (Recall that $\mathrm{Int}(S)$ consists of every $x \in X$ such that some open neighborhood of x is contained in S.)

Solution: We first show that $\mathrm{Cl}(S) \subseteq \mathrm{Int}(S) \cup \mathrm{Fr}(S)$. Let $x \in \mathrm{Cl}(S)$. If $x \in \mathrm{Int}(S)$, then some neighborhood of x is contained in S, hence $x \in S$, and since $S \subseteq \mathrm{Cl}(S)$, we see $x \in \mathrm{Cl}(S)$. If $x \notin \mathrm{Int}(S)$, then no neighborhood of x is contained in S, so every neighborhood of x meets S^c. But $x \in \mathrm{Cl}(S)$ implies that every neighborhood of x meets S, thus every neighborhood of x meets both S and S^c, hence $x \in \mathrm{Fr}(S)$. Therefore, if $x \in \mathrm{Cl}(S)$, then either $x \in \mathrm{Int}(S)$ or $x \in \mathrm{Fr}(S)$, so $x \in \mathrm{Int}(S) \cup \mathrm{Fr}(S)$, as we wanted to show.

Now we show that $\mathrm{Int}(S) \cup \mathrm{Fr}(S) \subseteq \mathrm{Cl}(S)$. As we saw above, if $x \in \mathrm{Int}(S)$, then $x \in S \subseteq \mathrm{Cl}(S)$. And if $x \in \mathrm{Fr}(S)$, then every neighborhood of x meets both S and S^c, but in particular every neighborhood of x meets S, so $x \in \mathrm{Cl}(S)$. Thus if $x \in \mathrm{Int}(S) \cup \mathrm{Fr}(S)$, then $x \in \mathrm{Cl}(S)$, so $\mathrm{Int}(S) \cup \mathrm{Fr}(S) \subseteq \mathrm{Cl}(S)$.

Problem 3: Let X be a topological space and S a connected subset. Show that $\mathrm{Cl}(S)$ is also connected.

Solution: We will prove the contrapositive. Assume $\mathrm{Cl}(S)$ is not connected. Then there are nonempty disjoint subsets $A, B \subset \mathrm{Cl}(S)$, open in the subspace topology (i.e., $A = U \cap \mathrm{Cl}(S)$ and $B = V \cap \mathrm{Cl}(S)$ for some open subsets U, V of X) with $A \cup B = \mathrm{Cl}(S)$. Let $A' = A \cap S$ and $B' = B \cap S$. Then $A' \cup B' = (A \cap S) \cup (B \cap S) = (A \cup B) \cap S = \mathrm{Cl}(S) \cap S = S$, $A' \cap B' \subseteq A \cap B \subseteq \varnothing$ (so $A' \cap B' = \varnothing$) and A' and B' are both open (in the subspace topology on S, since $A' = A \cap S = (U \cap \mathrm{Cl}(S) \cap S = U \cap S$ and $B' = B \cap S = (V \cap \mathrm{Cl}(S) \cap S = V \cap S)$. We just need to show that A' and B' are nonempty to conclude that S is not connected. The argument is the same for both; we will do it for A'.

Assume $A' = \varnothing$. Thus $S \cap A = \varnothing$, but $S \cap A = S \cap U \cap \mathrm{Cl}(S) = S \cap U$ (since $S \subseteq \mathrm{Cl}(S)$). Thus $S \cap U = \varnothing$, so $S \subseteq U^c$, so $\mathrm{Cl}(S) \subseteq U^c$, so $A = U \cap \mathrm{Cl}(S) = \varnothing$, which is a contradiction. Thus $A' \neq \varnothing$ (and similarly $B' \neq \varnothing$), so S is not connected.

Problem 4: Let X be a nonempty metric space with metric d. Let $S \subseteq X$. Show that S is bounded if and only if for every $y \in X$, there is an $r_y > 0$ such that $S \subseteq D_X(y, r_y)$.

Solution: By definition, if S is bounded, there is an $x \in X$ and an $r > 0$ such that $S \subseteq D_X(x, r)$. Now let $y \in X$. Pick $r_y = r + d(x, y)$. Let $z \in D_X(x, r)$. Then $d(z, y) \leq d(z, x) + d(x, y) < r + d(x, y) = r_y$, so $z \in D_X(y, r_y)$. Thus $S \subseteq D_X(x, r) \subseteq D_X(y, r_y)$.

Conversely, if for every $y \in X$, there is an $r_y > 0$ such that $S \subseteq D_X(y, r_y)$, then certainly there is some $x \in X$ such that there is an $r > 0$ such that $S \subseteq D_X(x, r)$.

Problem 5: Let $f : X \to Y$ be a continuous map of topological spaces. Let $S \subseteq Y$. Show that $\mathrm{Cl}(f^{-1}(S)) \subseteq f^{-1}(\mathrm{Cl}(S))$.

Solution: Since $S \subseteq \mathrm{Cl}(S)$, we have $f^{-1}(S) \subseteq f^{-1}(\mathrm{Cl}(S))$, but $\mathrm{Cl}(S)$ is closed and f is continuous, so $f^{-1}(\mathrm{Cl}(S))$ is closed, hence contains the closure of $f^{-1}(S)$; i.e., $\mathrm{Cl}(f^{-1}(S)) \subseteq f^{-1}(\mathrm{Cl}(S))$.

Here is an alternative proof. Let $x \in \mathrm{Cl}(f^{-1}(S))$ (so every open neighborhood of x meets $f^{-1}(S)$). We want to show that $x \in f^{-1}(\mathrm{Cl}(S))$; i.e., that $f(x) \in \mathrm{Cl}(S)$ and thus that every open neighborhood of $f(x)$ meets S. Let V be an open neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open neighborhood of x and hence meets $f^{-1}(S)$; i.e., $(f^{-1}(V)) \cap (f^{-1}(S)) \neq \varnothing$. Let $p \in (f^{-1}(V)) \cap (f^{-1}(S))$. Then $f(p) \in V \cap S$, so in particular $V \cap S \neq \varnothing$, as we wanted to show.

Problem 6: Let $f : X \to Y$ be a continuous map of topological spaces. Let $S \subseteq Y$. Show that $\mathrm{Cl}(f^{-1}(S)) = f^{-1}(\mathrm{Cl}(S))$ need not hold (give a specific example).

Solution: Let X be the reals with the discrete topology and let Y be the reals with the standard topology. Thus $X = Y$ as sets; let $f : X \to Y$ be the identity map. Since every subset of X is open, f is continuous. Let $S = (0, 1)$, so $\mathrm{Cl}(S) = [0, 1]$ and $f^{-1}(\mathrm{Cl}(S)) = [0, 1]$. However, $f^{-1}((0, 1)) = (0, 1)$ is closed in X so $\mathrm{Cl}(f^{-1}(S)) = \mathrm{Cl}(f^{-1}((0, 1))) = \mathrm{Cl}((0, 1)) = (0, 1)$.