Homework 3, due Thursday, September 13, 2012

Let X be a metric space with metric d. Let $S \subseteq X$. We say that S is *bounded* if for some $r > 0$ and $x \in X$ we have $S \subseteq D_X(x, r)$.

If X is a topological space and $S \subseteq X$, note that $\mathrm{Cl}(S) = S$ if S is closed. I.e., the closure of a closed set is the set itself. [We had two equivalent definitions of the closure $\mathrm{Cl}(S)$ of a set S. You can use whichever one you prefer. The first was that $\mathrm{Cl}(S)$ consists of all $x \in X$ such that every open neighborhood of x meets S. The second was that $\mathrm{Cl}(S)$ is the intersection of all closed subsets of X that contain S. Here is a proof that $\mathrm{Cl}(S) = S$ if S is closed, using definition 1: if $x \in S$ then every open neighborhood U of x clearly meets S (since x is in both U and S), and hence (by definition 1) $x \in \mathrm{Cl}(S)$. Thus $S \subseteq \mathrm{Cl}(S)$. And if $x \in \mathrm{Cl}(S)$, then every open neighborhood of x meets S (by definition 1). If $x \notin S$, then $x \in S^c$ and S^c is open (since S is closed), so S^c is an open neighborhood of x that does not meet S, which is a contradiction. Thus we must have $x \in S$, and hence $\mathrm{Cl}(S) \subseteq S$. Now here is a proof that $\mathrm{Cl}(S) = S$ if S is closed, using definition 2: by definition 2, $\mathrm{Cl}(S) = \cap_{C \in A} C$, where A is the set of all closed subsets of X that contain S, so $S \subseteq \mathrm{Cl}(S)$ since $\mathrm{Cl}(S)$ is an intersection of sets C each of which contains S. And $\mathrm{Cl}(S) \subseteq S$ since one of the sets C whose intersection is $\mathrm{Cl}(S)$ is S itself, since by hypothesis S is a closed subset of X which contains S.]

Do any 5 of the 6 problems. Each problem is worth 20 points. Solutions will be graded for correctness, clarity and style.

(1) Let X be a topological space and let $S \subseteq X$. Show that $\mathrm{Fr}(S)$ is closed. (Recall that $\mathrm{Fr}(S)$ consists of every $x \in X$ such that every open neighborhood of x meets both S and S^c.)

(2) Let X be a topological space and let $S \subseteq X$. Show that $\mathrm{Cl}(S) = \mathrm{Int}(S) \cup \mathrm{Fr}(S)$. (Recall that $\mathrm{Int}(S)$ consists of every $x \in X$ such that some open neighborhood of x is contained in S.)

(3) Let X be a topological space and S a connected subset. Show that $\mathrm{Cl}(S)$ is also connected.

(4) Let X be a nonempty metric space with metric d. Let $S \subseteq X$. Show that S is bounded if and only if for every $y \in X$, there is an $r_y > 0$ such that $S \subseteq D_X(y, r_y)$.

(5) Let $f : X \to Y$ be a continuous map of topological spaces. Let $S \subseteq Y$. Show that $\mathrm{Cl}(f^{-1}(S)) \subseteq f^{-1}(\mathrm{Cl}(S))$.

(6) Let $f : X \to Y$ be a continuous map of topological spaces. Let $S \subseteq Y$. Show that $\mathrm{Cl}(f^{-1}(S)) = f^{-1}(\mathrm{Cl}(S))$ need not hold (give a specific example).