Homework 1 Solutions

Let X and Y be sets and let $f : X \to Y$ be a mapping. Let $A, B \subseteq X$ and $C, D \subseteq Y$ be subsets, and let $A_i \subseteq X$, $i \in I$, be a family of subsets, indexed by some set I.

Remember: the goal in writing proofs is not only to be right, but to be understood. Never turn in an assignment without proofreading it. (As Truman Capote said in reference to the Beat writers, "That isn't writing at all, it's typing.") And when you proofread your proofs, check not only for correctness but ask yourself: is it clear? Is it unambiguous? Could I say the same thing more simply? Could I shorten it and still be clear? Would a few extra words be helpful? Also remember that proof writing is a form of writing and thus requires sentences. Proofs with no words are hard to read. A specific general rule is never to begin a sentence with a symbol. So every proof should have at least one sentence and thus at least one word!

Each problem is worth 20 points. Solutions will be graded for correctness, clarity and style.

(1) Prove that $(\cup_{i\in I} A_i)^c = \cap_{i\in I} A_i^c$. (If S is a subset of a set T, recall that S^c denotes the complement of S in T; i.e., $S^c = T \setminus S$. Note that the notation S^c is ambiguous if S is a subset of two different sets.)

Solution: First verify $(\cup_{i\in I} A_i)^c \subseteq \cap_{i\in I} A_i^c$. Let $a \in (\cup_{i\in I} A_i)^c$. Thus $a \notin \cup_{i\in I} A_i$, so a cannot be in any of the sets A_i; i.e., for all $i \in I$, we have $a \notin A_i$, hence $a \in A_i^c$ for all $i \in I$. Thus $a \in \cap_{i\in I} A_i^c$.

Now verify $(\cup_{i\in I} A_i)^c \supseteq \cap_{i\in I} A_i^c$. Let $a \in \cap_{i\in I} A_i^c$. Thus $a \in A_i^c$ for all $i \in I$, hence $a \notin A_i$ for all $i \in I$, so $a \notin \cup_{i\in I} A_i$, hence $a \in (\cup_{i\in I} A_i)^c$.

(2) Prove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

Solution: First verify $f^{-1}(C\cup D) \subseteq f^{-1}(C)\cup f^{-1}(D)$. Let $x \in f^{-1}(C\cup D)$. Thus $f(x) \in C\cup D$, so either $f(x) \in C$ or $f(x) \in D$, hence either $x \in f^{-1}(C)$ or $x \in f^{-1}(D)$, so $x \in f^{-1}(C) \cup f^{-1}(D)$.

Now verify $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$. Let $x \in f^{-1}(C) \cup f^{-1}(D)$. Thus either $x \in f^{-1}(C)$ or $x \in f^{-1}(D)$, so either $f(x) \in C$ or $f(x) \in D$, so $f(x) \in C \cup D$, hence $x \in f^{-1}(C \cup D)$.

(3) Is $f(A \cup B) = f(A) \cup f(B)$ always true? If so, prove it. If not, give a specific example for which equality does not hold.

Solution: Yes, $f(A \cup B) = f(A) \cup f(B)$ is always true. We first verify $f(A \cup B) \subseteq f(A) \cup f(B)$. Let $y \in f(A \cup B)$. Thus there is an $x \in A \cup B$ such that $f(x) = y$. Either $x \in A$ or $x \in B$, so either $f(x) \in f(A)$ or $f(x) \in f(B)$, hence $y = f(x) \in f(A) \cup f(B)$.

Now we verify $f(A\cup B) \supseteq f(A)\cup f(B)$. Let $y \in f(A)\cup f(B)$. Then either $y \in f(A)$ (in which case $y = f(a)$ for some $a \in A$) or $y \in f(B)$ (in which case $y = f(b)$ for some $b \in B$). But $f(a) \in f(A\cup B)$ and $f(b) \in f(A \cup B)$, so either way $y \in f(A \cup B)$.

(4) Is $f(A \cap B) = f(A) \cap f(B)$ always true? If so, prove it. If not, give a specific example for which equality does not hold.

Solution: It is not always true that $f(A \cap B) = f(A) \cap f(B)$. Let $X = Y = \mathbf{R}$ be the reals, and let f be $f(x) = x^2$. Consider the intervals $A = [-1, 0]$ and let $B = [0, 1]$. Then $A \cap B = \{0\}$, so $f(A \cap B) = \{0\}$, but $f(A) = [0, 1] = f(B)$, so $f(A) \cap f(B) = [0, 1]$.

(5) Which of $f(f^{-1}(C)) \subseteq C$, $f(f^{-1}(C)) = C$, and $f(f^{-1}(C)) \supseteq C$ is always true? Prove those which are true and give specific examples showing the others can fail.

Solution: It is always true that $f(f^{-1}(C)) \subseteq C$, but $f(f^{-1}(C)) \supseteq C$ and hence $f(f^{-1}(C)) = C$ can fail. For example, let f be as in the solution to [4], so $f(x) = x^2$ and let $C = [-1, 1]$. Then $f^{-1}(C) = [-1, 1] = C$ and $f(C) = [0, 1] \subsetneq C$, so $f(f^{-1}(C)) \subsetneq C$, hence $f(f^{-1}(C)) \supseteq C$ does not hold.

Now we show $f(f^{-1}(C)) \subseteq C$ always holds. Let $c \in f(f^{-1}(C))$. Then $c = f(d)$ for some $d \in f^{-1}(C)$, and $f(d) \in C$; i.e., $c = f(d) \in C$, as we wanted to show.