Homework 1, due Tuesday, August 28, 2012

Let X and Y be sets and let $f : X \to Y$ be a mapping. Let $A, B \subseteq X$ and $C, D \subseteq Y$ be subsets, and let $A_i \subseteq X$, $i \in I$, be a family of subsets, indexed by some set I.

Remember: the goal in writing proofs is not only to be right, but to be understood. Never turn in an assignment without proofreading it. (As Truman Capote said in reference to the Beat writers, "That isn't writing at all, it's typing.") And when you proofread your proofs, check not only for correctness but ask yourself: is it clear? Is it unambiguous? Could I say the same thing more simply? Could I shorten it and still be clear? Would a few extra words be helpful? Also remember that proof writing is a form of writing and thus requires sentences. Proofs with no words are hard to read. A specific general rule is never to begin a sentence with a symbol. So every proof should have at least one sentence and thus at least one word!

Each problem is worth 20 points. Solutions will be graded for correctness, clarity and style.

(1) Prove that $(\cup_{i\in I} A_i)^c = \cap_{i\in I} A_i^c$. (If S is a subset of a set T, recall that S^c denotes the complement of S in T; i.e., $S^c = T \setminus S$. Note that the notation S^c is ambiguous if S is a subset of two different sets.)

(2) Prove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

(3) Is $f(A \cup B) = f(A) \cup f(B)$ always true? If so, prove it. If not, give a specific example for which equality does not hold.

(4) Is $f(A \cap B) = f(A) \cap f(B)$ always true? If so, prove it. If not, give a specific example for which equality does not hold.

(5) Which of $f(f^{-1}(C)) \subseteq C$, $f(f^{-1}(C)) = C$, and $f(f^{-1}(C)) \supseteq C$ is always true? Prove those which are true and give specific examples showing the others can fail.