M445/845, Exam 3, due Friday, December 6, 2013

Instructions: If you are a 445 student, do any three problems. If you are an 845 student, do any four problems. Although the due date is December 6, if you'd like more time you can have an extension, but you must turn in your paper by the time of the final exam time slot, 3:30 p.m., Thursday, December 19. Since I'm dropping one exam grade, this exam cannot drop your average. However, I will count it either as an extra homework or as an exam, depending on which helps your grade more. You may discuss the problems with others, but write up your solutions independently.

(1) Determine whether the following statement is True or False, and justify your answer (i.e., prove it if it is true or give a specific counterexample if it is false): If $f(x,y) = ax^2 + bxy + cy^2$ is an integral positive definite binary quadratic form, then the least positive value of $f(i,j)$ where i and j are integers occurs either for $i = 0$ and j is either 1 or -1, or for $j = 0$ and i is either 1 or -1.

(2) For each of the following binary quadratic forms $f(x,y)$, determine whether or not there is a nontrivial integer solution to $f(x,y) = 0$ (i.e., a solution where x and y are integers but not both 0). If so, find such a solution. If not, justify why no such solution exists.

(a) $f(x,y) = x^2 + 4xy + y^2$

(b) $f(x,y) = 3x^2 + 4xy + y^2$

(3) Let $f(x,y)$ be an integral binary quadratic form. If $f(x,y) = 0$ has a nontrivial solution, show it has infinitely many.

(4) Prove for each integer d congruent modulo 4 either to 0 or 1 that there are infinitely many integral binary quadratic forms of discriminant d. [Hint: if you find one, consider a change of basis in which $x \mapsto x + ty$ and $y \mapsto y$, where t is any integer.]

(5) Determine whether or not $2x^2 + 7xy + y^2$ represents 47. If so, find a solution over the integers to $2x^2 + 7xy + y^2 = 47$.

(6) Determine whether or not $7x^2 + 6xy - y^2$ represents 47. If so, find a solution over the integers to $7x^2 + 6xy - y^2 = 47$.

(7) Determine whether or not $7x^2 + 2xy + 23y^2$ represents 47. If so, find all solutions over the integers to $7x^2 + 2xy + 23y^2 = 47$ and justify why you have found all of them.