M104, Practice Exam 2 Solutions

Exam 2 is on Wednesday, February 27, 2013 in class in our usual classroom. The problems on the exam will be modeled on the ones here. Please work out the problems here in preparation for a class discussion on Monday, February 25. For each problem, you must show enough work to justify your answer.

(1) (30 points) Find $f'(x)$ for each of the following functions $f(x)$.

(a) $f(x) = \ln(x^2 + x)$: Answer (chain rule and rule for logs):

$$f'(x) = \frac{(x^2 + x)'}{x^2 + x} = \frac{2x + 1}{x^2 + x}$$

(b) $f(x) = \dfrac{x^2 + x + 3}{x^3 - 7x + 1}$: Answer (quotient rule):

$$\begin{aligned} f'(x) &= \frac{(x^2 + x + 3)'(x^3 - 7x + 1) - (x^2 + x + 3)(x^3 - 7x + 1)'}{(x^3 - 7x + 1)^2} \\ &= \frac{(2x + 1)(x^3 - 7x + 1) - (x^2 + x + 3)(3x^2 - 7)}{(x^3 - 7x + 1)^2} \end{aligned}$$

(c) $f(x) = (x^3 + 3x^2 - x + 7)^5$: Answer (chain rule):

$$f'(x) = 5(x^3 + 3x^2 - x + 7)^4(x^3 + 3x^2 - x + 7)' = 5(x^3 + 3x^2 - x + 7)^4(3x^2 + 6x - 1)$$

(d) $f(x) = \ln(x)e^x$: Answer (product rule and rules for logs and exponentials):

$$f'(x) = (\ln(x))'e^x + \ln(x)(e^x)' = \frac{1}{x}e^x + \ln(x)e^x$$

(e) $f(x) = (x^3 + 3x^2 - x + 7)(6x^5 + 4x^4 - x^3 + 8x^2 - 12x + 13)$: Answer (product rule):

$$\begin{aligned} f'(x) =&(x^3 + 3x^2 - x + 7)'(6x^5 + 4x^4 - x^3 + 8x^2 - 12x + 13) + \\ &(x^3 + 3x^2 - x + 7)(6x^5 + 4x^4 - x^3 + 8x^2 - 12x + 13)' \\ =&(3x^2 + 6x - 1)(6x^5 + 4x^4 - x^3 + 8x^2 - 12x + 13) + \\ &(x^3 + 3x^2 - x + 7)(30x^4 + 16x^3 - 3x^2 + 16x - 12) \end{aligned}$$

(2) (24 points) Find the account balance A for a deposit of $P = \$125$ after $t = 10$ years, in each of the following scenarios.

(a) Simple interest at an annual rate of $r = 1\%$: $A = P + Prt = P(1 + rt) = 125(1 + 0.01(10)) = 125(1.1) = \$137.5$.

(b) Compound interest at an annual rate of $r = 1\%$ compounded 4 times per year: $A = P(1 + \frac{r}{m})^{mt} = 125(1 + \frac{0.01}{4})^{4(10)} = 125(1.0025)^{40} = \$138.13$.

(c) Continuously compounded interest at an annual rate of $r = 1\%$: $A = Pe^{rt} = 125e^{0.01(10)} = 125e^{0.1} = \$138.15$.

(d) The doubling time is 70 years: $A = P2^{t/70} = 125(2^{10/70}) = \$138.01$.

(3) (10 points) Which investment is better, one earning 5% interest compounded daily or one whose doubling time is 14 years.

At 5% compounded daily, \$1 becomes $(1 + \frac{.05}{365})^{365} = 1.05127$ after 1 year, while with a doubling time of 14 years, \$1 becomes $2^{1/14} = 1.05076$ after 1 year. Thus 5% compounded daily is better.

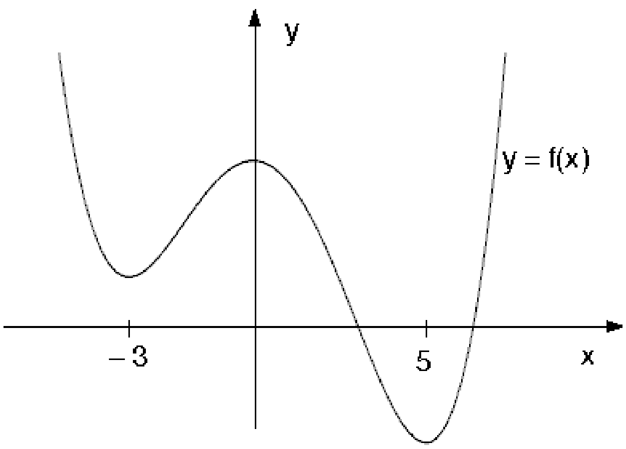
(4) (10 points) Determine the open intervals on which $f(x) = x^4 - 4x^3$ is increasing and the open intervals on which $f(x) = x^4 - 4x^3$ is decreasing.

We have $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ so $f'(x) = 0$ for $x = 0$ and $x = 3$. Now we must determine the sign of $f'(x)$ when it is not zero. Since $f'(-1) < 0$, $f'(1) < 0$ and $f'(4) > 0$, we see $f'(x) < 0$ when $x < 0$, $f'(x) < 0$ when $0 < x < 3$ and $f'(x) > 0$ when $x > 3$. Thus $f(x)$ is decreasing when $x < 0$, $f(x)$ is decreasing when $0 < x < 3$ and $f(x)$ is increasing when $x > 3$.

(5) (10 points) Determine the open intervals on which $f(x)$ is increasing and the open intervals on which $f(x)$ is decreasing if $f'(x) = x(x-2)(x-4)$.

We see $f'(x) = 0$ for $x = 0, 2, 4$. Now we must determine the sign of $f'(x)$ when it is not zero. Since $f'(-1) < 0$, $f'(1) > 0$, $f'(3) < 0$ and $f'(5) > 0$, we see that $f'(x) < 0$ for $x < 0$, $f'(x) > 0$ for $0 < x < 2$, $f'(x) < 0$ for $2 < x < 4$ and $f'(x) > 0$ for $x > 4$. Thus $f(x)$ is decreasing for $x < 0$, increasing for $0 < x < 2$, decreasing for $2 < x < 4$ and increasing for $x > 4$.

(6) (8 points) Determine the open intervals on which $f(x)$ is increasing and the open intervals on which $f(x)$ is decreasing for the function $y = f(x)$ in the graph below.



From the graph we see that $f(x)$ is decreasing for $x < -3$, increasing for $-3 < x < 0$, decreasing for $0 < x < 5$ and increasing for $x > 5$.

(7) (8 points) Consider $y = f(x)$, where $f(x)$ has a vertical asymptote at $x = 1$, a horizontal asymptote at $y = 0$, $f'(x) = 0$ only at $x = 0$, $f(0) = 1$ and $f'(-1) < 0$, $f'(\frac{1}{2}) > 0$ and $f'(2) < 0$.

(a) What are the critical values of $f(x)$? Answer: $x = 0$ (because $f'(x) = 0$) and $x = 1$ (because of the vertical asymptote).

(b) Determine the open intervals on which $f(x)$ is increasing and the open intervals on which $f(x)$ is decreasing.

Since $f'(-1) < 0$, we know that $f(x)$ is decreasing for $x < 0$. Since $f'(\frac{1}{2}) > 0$, we know $f(x)$ is increasing for $0 < x < 1$. Since $f'(2) < 0$, we know $f(x)$ is decreasing for $x > 1$.

(c) Sketch a graph of $y = f(x)$ taking into account all of the given information.