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On the Asymptotic Behavior of Some Population Models*

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0. INTRODUCTION

In this paper we consider the following nonautonomous Kolmogorov population system

$$u_i' = u_i F_i(t, u), \qquad u = (u_1, ..., u_n), \qquad 1 \leqslant i \leqslant n, \tag{0.1}$$

where $F: \mathbb{R} \times \mathbb{R}^n_+ \to \mathbb{R}^n$ is a continuous function and $\mathbb{R}^n_+$ is the nonnegative orthant, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n: x_i \geqslant 0, 1 \leqslant i \leqslant n\}$. These systems arise naturally in population biology. See [7] for the autonomous case and [10] for the periodic case.

In order to apply the usual theorems about O.D.E., we shall assume the following restriction on F:

(H_1) F is locally Lipschitz. That is: for all (t_0, x_0) in $\mathbb{R} \times \mathbb{R}^n_+$ there exists a neighborhood N of this point, and a positive constant M, such that $\|F(t, x) - F(t, y)\| \leqslant M\|x - y\|$ for all (t, x), (t, y) in N. Here, $\|x\| = |x_1| + \cdots + |x_n|$ if $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

(H_2) F is bounded in $\mathbb{R} \times K$ for each compact subset K of $\mathbb{R}^n$.

Assume that the partial derivatives $(\partial F_i/\partial x_j)(t, x)$ are defined and continuous for all t in $\mathbb{R}$ and $x > 0$, and suppose that there are positive constants $m, c_1, ..., c_n$ such that

$$-c_i \frac{\partial F_i}{\partial x_i}(t, x) \geqslant m + \sum_{j \in J_i} c_j \left| \frac{\partial F_j}{\partial x_i}(t, x) \right|, \qquad 1 \leqslant i \leqslant n, \tag{0.2}$$

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where, from now on, $J_i = \{1, ..., i-1, i+1, ..., n\}$. If the system (0.1) has a positive solution $v = (v_1, ..., v_n)$, defined and bounded in $[0, \infty)$, we shall prove that:

(a) All solutions u of (0.1) with $u(0) > 0$, are defined on $[0, \infty)$ and

$$u(t) - v(t) \to 0 \qquad \text{as} \quad t \to \infty. \tag{0.3}$$

(b) System (0.1) has at most one solution u^0, defined on $\mathbb{R}$, whose components are bounded above and below by positive constants.

(c) If F is almost periodic, uniformly on compact subsets of $\mathbb{R}^n_+$, and $v_1, ..., v_n$ are bounded below by positive constants on $[0, \infty)$, then system (0.1) has an almost periodic solution, whose components are bounded, above and below, by positive constants. A parallel result holds in the periodic case.

(d) If F is T-periodic with respect to a time variable t, then system (0.1) has a nonnegative T-periodic solution u^0, such that $u(t) - u^0(t) \to 0$ as $t \to \infty$, for any positive solution u of (0.1).

These results have the advantage that we do not assume any sign condition on $\partial F_i/\partial x_j$ for $i \neq j$. So, we can study simultaneously several population models: competing species, predator–prey, mutualism, etc... . See [8, p. 36; 10].

Condition (0.2) is quite restrictive, but it can be applied successfully when $F_i(t, u)$ has the form

$$F_i(t, u) = a_i(t) - \sum_{j=1}^{n} b_{ij}(t) u_j \tag{0.4}$$

and $a_i, b_{ij}: \mathbb{R} \to \mathrm{R}$ are bounded continuous functions. For example, assume $a_{iL} > 0$, $b_{ijL} > 0$, and

$$a_{iL} > \sum_{j \in J_i} b_{ijM} a_{jM}/b_{jjL}, \tag{0.5}$$

where, in the next, $g_L(g_M) = \inf(\sup)\{g(t): t \in \mathbb{R}\}$ for each bounded function $g: \mathbb{R} \to \mathbb{R}$. We shall prove that in this case, F satisfies (0.2), and (0.1) has a solution v as above. Thus, we improve the main results in [1, 2, 5, 6].

Remark. If $n = 2$ and F_i is given by (0.4), then (0.2) is implied by: $b_{iiL} > 0$, $|b_{2i}|_L > 0$, and

$$\sup(|b_{12}|/b_{22}) < \inf(b_{11}/|b_{21}|). \tag{0.6}$$

Remark. Let $\alpha_1, ..., \alpha_n: \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions such that the derivatives $\alpha'_1, ..., \alpha'_n$ are bounded below by positive constants. Then the assertions above remain true if we replace u_j by $\alpha_j(u_j)$ in (0.4).

To end this paper, we prove a stable coexistence theorem for the periodic predator–prey model.

1. The Main Results

We begin with some notations. Given $x=(x_1, ..., x_n)$ in $\mathbb{R}^n$, we write $x>0$ $(x \geqslant 0)$ if $x_i>0$ $(x_i \geqslant 0)$ for $1 \leqslant i \leqslant n$. We also define $\|x\| = |x_1| + \cdots + |x_n|$.

The maximal domain of a solution u of (0.1), is denoted by $\mathrm{dom}(u)$. Notice that, if $u(\tau)>0$ for some τ, then $u(t)>0$ for all t in $\mathrm{dom}(u)$. In this case we say that u is a positive solution to (0.1).

Since we are interested in the almost periodic case, we must study system (0.1) in the nondifferentiable case. Thus, we shall assume that there are positive constants $c_1, ..., c_n$ and a continuous function $m: \mathbb{R} \to [0, \infty)$ such that

$$\begin{aligned} c_i[F_i(t, x_h^i) - F_i(t, x_h^{i-1})] + m(t)\, h_i \\ \leqslant -\sum_{j \in J_i} c_j |F_j(t, x_h^i) - F_j(t, x_j^{i-1})| \end{aligned} \tag{1.1}$$

for t in $\mathbb{R}$, $1 \leqslant i \leqslant n$, $x=(x_1, ..., x_n)>0$, $h=(h_1, ..., h_n) \geqslant 0$, and $x_h^i := (x_1+h_1, ..., x_i+h_i, x_{i+1}, ..., x_n)$. Notice that (0.2) implies (1.1) if the partial derivatives $(\partial F_i/\partial x_j)(t, x)$ are defined and continuous for t in $\mathbb{R}$ and $x>0$.

Given positive solutions $u=(u_1, ..., u_n)$ and $v=(v_1, ..., v_n)$ to (0.1), we define

$$r(t) = r(t, u, v) = \sum_{j=1}^{n} c_j |\ln(u_j(t)) - \ln(v_j(t))|. \tag{1.2}$$

1.1. Theorem. *If* (1.1) *holds and* $I := \mathrm{dom}(u) \cap \mathrm{dom}(v) \neq \varnothing$, *then there exists a countable subset* N *of* I *such that* r *is differentiable on* $J := I \backslash N$ *and*

$$r'(t) \leqslant m(t) \|u(t) - v(t)\| \qquad \textit{for} \quad t \textit{ in } J. \tag{1.3}$$

Proof. We begin with the following remark. Let $f: (a, b) \to \mathbb{R}$ be differentiable and define $D = \{t: a<t<b, f(t)=0 \neq f'(t)\}$ and $g(t) = |f(t)|$. Then g is differentiable on $I \backslash D$ and D is countable, since D is discrete. From this, there exists a countable subset N of I such that $|\ln(u_j/v_j)|$ is differentiable on $J := I \backslash N$ for $1 \leqslant j \leqslant n$.

Let us fix t in J and define $S_+ = \{j: u_j(t) > v_j(t)\}$, $S_- = \{j: u_j(t) < v_j(t)\}$, $S_0 = \{j: u_j(t) = v_j(t)\}$, and $S = S_+ \cup S_-$. Notice that $S \neq \varnothing$ and $(d/dt)|\ln(u_j/v_j)| = 0$ in t, for all j in S_0. Consequently,

$$\begin{aligned} r'(t) = \sum_{j \in S_+} c_j [F_j(t, u(t)) - F_j(t, v(t))] \\ - \sum_{j \in S_-} c_j [F_j(t, u(t)) - F_j(t, v(t))]. \end{aligned}$$

Now, let us define x, h, k in $\mathbb{R}^n$ and Δ_{ij}^p in $\mathbb{R}$ by: $x_j = v_j(t)$ if $j \in S_+$; $x_j = u_j(t)$ if $j \in S_0 \cup S_-$; $x=(x_1, ..., x_n)$; $h = u(t) - x$; $k = v(t) - x$ and $\Delta_{ij}^p = F_j(t, x_p^i) - F_j(t, x_p^{i-1})$ for $p = h, k$. Since $F_j(t, u(t)) - F_j(t, v(t)) = F_j(t, x+h) - F_j(t, x) - [F_j(t, x+k) - F_j(t, x)]$, then

$$F_j(t, u(t)) - F_j(t, v(t)) = \sum_{i \in S_+} \Delta_{ji}^h - \sum_{i \in S_-} \Delta_{ji}^k.$$

Notice that $h_i = 0$ for $i \in S_- \cup S_0$ and hence $\Delta_{ji}^h = 0$ for i in $S_0 \cup S_-$, since $x_h^i = x_h^{i-1}$. Analogously, $\Delta_{ji}^k = 0$ if $i \in S_0 \cup S_+$.

From this,

$$\begin{aligned} r'(t) &= \sum_{j \in S_+} c_j \left[\sum_{i \in S_+} \Delta_{ji}^h - \sum_{i \in S_-} \Delta_{ji}^k \right] - \sum_{j \in S_-} c_j \left[\sum_{i \in S_+} \Delta_{ji}^h - \sum_{i \in S_-} \Delta_{ji}^k \right] \\ &= \sum_{i \in S_+} \left[\sum_{j \in S_+} c_j \Delta_{ji}^h - \sum_{j \in S_-} c_j \Delta_{ji}^k \right] - \sum_{i \in S_-} \left[\sum_{j \in S_+} c_j \Delta_{ji}^h - \sum_{j \in S_-} c_j \Delta_{ji}^k \right] \\ &= \sum_{i \in S_+} \lambda_{ih} - \sum_{i \in S_-} \lambda_{ik}, \end{aligned}$$

where

$$\lambda_{ip} = \sum_{j \in S_+} c_j \Delta_{ji}^p - \sum_{j \in S_-} c_j \Delta_{ji}^p, \qquad \text{for} \quad p = h, k.$$

For i in S_+, we have (see (1.1))

$$\lambda_{ih} \leqslant c_i \Delta_{ii}^h + \sum_{j \in J_i} c_j |\Delta_{ji}^h| \leqslant -m(t)\, h_i.$$

Analogously, $\lambda_{ik} \geqslant m(t)\, k_i$ for all i in S_-, and the proof is complete.

In the following, C_+ denotes the set of all bounded continuous functions $g: \mathbb{R} \to \mathbb{R}$ such that $g_L > 0$. We also assume hypotheses (H_1)–(H_2), which we have stated in the Introduction.

1.2. THEOREM. *Assume* (1.1), *and suppose that* (0.1) *has a positive solution* $v = (v_1, ..., v_n)$ *defined in* $[t_0, \infty)$, *for some* t_0, *such that*

$$M := \sup\{v_i(t): t \geqslant t_0,\ 1 \leqslant i \leqslant n\} < +\infty. \tag{1.4}$$

If $u = (u_1, ..., u_n)$ *is a positive solution to* (0.1) *such that* $I := \mathrm{dom}(u) \cap \mathrm{dom}(v) \neq \varnothing$, *then* u *is defined and bounded in* $[t_*, \infty)$, *if* $t_* \in I$. *We also have the following facts*:

(a) *If* $m > 0$ *is constant then* (0.3) *holds.*

(b) *If*

$$\varepsilon := \inf\{v_i(t): t \geqslant t_0,\ 1 \leqslant i \leqslant n\} > 0 \tag{1.5}$$

then there exist positive constants λ, μ *such that*

$$\|u(t) - v(t)\| \leqslant \lambda \|u(t_1) - v(t_1)\| \exp\left(-\mu \int_{t_1}^{t} m(s)\, ds\right) \tag{1.6}$$

for $t_* \leqslant t_1 \leqslant t$. *In particular,* (0.3) *holds if*

$$\int_0^\infty m(s)\, ds = +\infty. \tag{1.7}$$

(c) *The problem*

$$u_i' = u_i F_i(t, u), \qquad u_i \in C_+,\ 1 \leqslant i \leqslant n \tag{1.8}$$

has at most one solution if

$$\int_{-\infty}^{0} m(s)\, ds = +\infty. \tag{1.9}$$

Proof. From (1.3), we know that r is a decreasing function on I. In particular, $r(t) \leqslant r(t_*)$ for all $t \geqslant t_*$, $t \in I$. Hence, there are positive constants p, q such that

$$q v_j(t) \leqslant u_j(t) \leqslant p v_j(t), \qquad 1 \leqslant j \leqslant n,\ t \in I \cap [t_*, \infty). \tag{1.10}$$

Thus, u is bounded in $I \cap [t_*, \infty)$, and by (H_2) and an elementary theorem about continuation of solutions, we know that u is defined in $[t_*, \infty)$.

From (1.3) we also have

$$\int_{t_*}^{t} \|u(s) - v(s)\|\, ds \leqslant (1/m)[r(t) - r(t_*)], \qquad \text{for} \quad t \geqslant t_*$$

and hence

$$\int_0^\infty \|u(s) - v(s)\|\, ds < +\infty. \tag{1.11}$$

But u, v are bounded in $[t_*, \infty)$ and by (H_2), the same holds for u', v'. From this and (1.11) we get $\|u(t) - v(t)\| \to 0$ as $t \to \infty$ and so the proof of (a) is complete.

(b) By (1.4)–(1.5)–(1.10) and the usual mean value theorem, there are positive constants α_0, α_1 such that

$$\frac{1}{\alpha_0} |u_j(t) - v_j(t)| \leqslant |\ln(u_j(t)) - \ln(v_j(t))|$$

$$\leqslant \frac{1}{\alpha_1} |u_j(t) - v_j(t)|,$$

for $t \geqslant t_*$ and $1 \leqslant i \leqslant n$. From this, there are positive constants α, α_2 such that

$$\alpha_2 \|u(t) - v(t)\| \leqslant r(t) \leqslant \alpha^{-1} \|u(t) - v(t)\| \tag{1.12}$$

for $t \geqslant t_*$. Thus, $r'(t) \leqslant -\alpha m(t)\, r(t)$ for $t \geqslant t_*$ $(t \in J)$, and hence

$$r(t) \leqslant r(t_1) \exp\left(-\int_{t_1}^{t} \alpha m(s)\, ds\right), \qquad \text{if} \quad t \geqslant t_1 \geqslant t_*.$$

The proof of (0.6) follows now from (1.12) and so the proof of (b) is complete.

Assume now that u, v are solutions to (1.8). From the arguments above we conclude that (0.6) holds if $t_1 \leqslant t$. Hence

$$\lambda \|u(t) - v(t)\| \geqslant \|u(0) - v(0)\| \exp\left(\mu \int_t^0 m(s)\, ds\right)$$

for $t \leqslant 0$ and then $u(0) = v(0)$, since (1.9) holds and u and v are bounded. Thus, the proof is complete.

1.3. COROLLARY. *Suppose that* F *is periodic in time, with period* $T > 0$, *and assume that there is a solution* v *of* (0.1), *defined in* $[t_0, \infty)$, *which satisfies* (1.4)–(1.5). *If* (1.1) *and* (1.7) *hold, then system* (0.1) *has exactly one* T-*periodic positive solution.*

Proof. Let us define $v^k(t) = v(t + kT)$ for all integers $k \geqslant 1$ and $t \geqslant t_0 - kT$, and choose a subsequence $\{v^m(0)\}$ of $\{v^k(0)\}$ such that

$v^m(0) \to x \in \mathbb{R}^n$ as $m \to \infty$. Since $\varepsilon \leqslant v_i^k(t) \leqslant M$ for all components of v^k, it is easy to prove (see Lemma 1 of [1]) that the solution u of (0.1), having the initial condition $u(0) = x$, is defined on $\mathbb{R}$ and $\varepsilon \leqslant u_i \leqslant M$ for all components u_i of u.

On the other hand, by Theorem 1.2 we know that $v^1(t) - v(t) \to 0$ as $t \to \infty$, and so, $v^{m+1}(0) \to x$ as $m \to \infty$. Thus $v^m(T) \to x$ as $m \to \infty$ and then $u(0) = u(T)$. Therefore, u is T-periodic and the proof follows from Theorem 1.2.

1.4. THEOREM. *Suppose that* (0.1) *has a positive solution* v *defined in* $[t_0, \infty)$, *which satisfies* (1.4)–(1.5) *and assume that* $F(t, x)$ *is almost periodic uniformly for* x *in* $[\varepsilon, M]^n$.

Assume further that for all $x^0 > 0$ *in* $\mathbb{R}^n$, *there exists an open subset* U *of* $\mathbb{R}^n$ *containing* x^0, *and a constant* $\lambda > 0$ *such that* $\|F(t, x) - F(t, y)\| \leqslant \lambda \|x - y\|$ *for all* $x, y \in U$ *and* t *in* $\mathbb{R}$.

If (1.1) *holds, then problem* (1.8) *has exactly one solution* u *and* u *is almost periodic and* mod $u \subseteq$ mod F.

Proof. Let (t_k) be a sequence of $\mathbb{R}$. Without loss of generality, we can assume that there exists a continuous function $G\colon \mathbb{R} \times [\varepsilon, M]^n \to \mathbb{R}^n$ such that

$$F(t + t_k, x) \to G(t, x) \qquad \text{as} \quad k \to \infty \text{ unif. on } \mathbb{R} \times K.$$

In particular, G is locally Lipschitz in the sense of (H_1) and satisfies (1.1). Thus, by Theorem 1.2, the problem

$$u_i' = u_i G_i(t, u), \qquad u_i \in C_+,\ 1 \leqslant i \leqslant n \tag{1.13}$$

has at most one solution belonging to $K := [\varepsilon, M]^n$.

Now let us define $v^k(t) = v(t + t_k)$ for all integers $k \geqslant 1$ and $t \geqslant t_0 - t_k$. Then v^k is a solution to the system

$$u_i' = u_i F_i(t + t_k, u). \qquad 1 \leqslant i \leqslant n$$

belonging to K. In particular, we can assume that $v^k(0) \to x \in K$ as $k \to \infty$. From this, the solution u to the initial value problem $u_i' = u_i G_i(t, u)$, $1 \leqslant i \leqslant n$, $u(0) = x$ is a solution to (1.13) and the proof follows from Theorem 10.1 of [4].

1.5. THEOREM. *Assume* (0.2) *and suppose that* (0.1) *has a positive solution* v, *defined and bounded in* $[t_0, \infty)$. *If* F *is* T-*periodic with respect to time variable* t *then there exists a (unique)* T-*periodic nonnegative solution* u^0 *of* (0.1) *such that* $u(t) - u^0(t) \to 0$ *as* $t \to \infty$, *for any positive solution* u *to* (0.1).

Proof. For each subset I of $I_n := \{1, 2, ..., n\}$ let $S(I)$ be the set of all T-periodic solutions u of (0.1) such that $u_i > 0$ (resp. $u_i = 0$) for all $i \in I$ (resp. $i \in I_n \backslash I$). Notice that $S(\varnothing)$ consists exactly of the trivial solution. Assume now that $I \neq \varnothing$. Without loss of generality, we can suppose that $I = \{1, ..., p\}$ for some $1 \leqslant p \leqslant n$. If $u = (u_1, ..., u_n) \in S(I)$, then $(u_1, ..., u_p)$ is a solution to the problem

$$u_i' = u_i F_i(t, u_1, ..., u_p, 0, ..., 0), \qquad u_i \in C_+,\ 1 \leqslant i \leqslant p.$$

From this, and Theorem 1.2, $S(I)$ has at most a point and so, the set S, of all nonnegative T-periodic solutions of (0.1), has at most 2^n points. Notice that the trivial solution ($u \equiv 0$) to (0.1), belongs to S.

Let u be a positive solution of (0.1). For all integers k, we define $u^k(t) = u(t + kT)$. Notice that, by Theorem 1.2(a), $v^k(t) - v(t) \to 0$ as $t \to \infty$. Let us fix t_* in dom(u); it is clear that $t_* \in \mathrm{dom}(v^k)$ for some k, and by Theorem 1.2, u is defined on $[t_*, \infty)$ and $u(t) - v^k(t) \to 0$ as $t \to \infty$. From this, $u(t) - v(t) \to 0$ as $t \to \infty$ and hence,

$$u(t) - w(t) \to 0 \qquad \text{as} \quad t \to \infty \tag{1.14}$$

for any positive solutions u, w to (0.1).

For each p in $\mathbb{R}^n_+$, let $u(t, p)$ be the solution to (0.1) given by $u(0, p) = p$, and let D be the subset of $\mathbb{R}^n_+$ consisting of all points p such that $u(t, p)$ is defined on $[0, T]$. We know that $p \in D$ if $p > 0$. Moreover, the set Fix(H), of all fixed points of the Poincaré map, $H\colon D \to \mathbb{R}^n$, $H(p) := u(T, p)$, is finite and nonempty. Notice that $\mathrm{Fix}(H) = \{u(0)\colon u \in S\}$.

Let us fix $p > 0$. By (1.14), $u(t + T, p) - u(t, p) \to 0$ as $t \to \infty$, and hence $H^k(p) - H^{k+1}(p) \to 0$ as $k \to \infty$. Since $\{H^k(p)\}$ is a bounded sequence, it is easy to prove:

Claim. Each subsequence of $\{H^k(p)\}$ has a convergent subsequence to a point $q \geqslant 0$. Moreover, $u(t, q)$ is defined on $\mathbb{R}$ and $H(q) = q$.

Let us write $\mathrm{Fix}(H) = \{p_1, ..., p_s\}$ and choose closed balls $B_1, ..., B_s$ about $p_1, ..., p_s$, respectively, such that $B_i \cap B_j = \varnothing$ if $i \neq j$. Let N_j (resp. N) be the set of all integers $k \geqslant 1$ such that $H^k(p) \in B_j$ (resp. $k \notin N_j$, for any j). From the above claim, N is finite and so there exists j such that N_j is infinite. Let us write $N_j = \{n_1 < n_2 < \cdots\}$. By the above claim, $\{H^m(p)\colon m \in N_j\}$ converges to p_j, and then, the same holds for $\{H^{m+1}(p)\colon m \in N_j\}$.

Let U be an open subset of $\mathbb{R}^n$ containing B_j, such that $U \cap B_i = \varnothing$, for $i \neq j$, and $H^k(p) \notin U$ if $k \in N$. Then there exists m_0 in N_j such that $H^m(p)$, $H^{m+1}(p) \in U$ if $m \geqslant m_0$ and $m \in N_j$. From this, $H^m(p)$, $H^{m+1}(p) \in B_j$ for $m \geqslant m_0$, $m \in N_j$. That is, $m + 1$ belongs to N_j if $m \in N_j$ and $m \geqslant m_0$. Consequently, N_j contains all integers $m \geqslant m_0$ and then $H^m(p) \to p_j$ as $m \to \infty$. Therefore, $u(t, p) - u(t, p_j) \to 0$ as $t \to \infty$.

By (1.14), $u(t, p)-u(t) \to 0$ as $t \to \infty$, for any positive solution u of (0.1). Thus, the proof is complete, since $u^0(t) \equiv u(t, p_j)$ is T-periodic and non-negative.

1.6. REMARK. Let u^0 be the solution given by Theorem 1.5, and assume that $u_i^0=0$ for some i. Then $\int_0^T F_i(t, u^0(t))\, dt \leqslant 0$.

Proof. Let us fix a positive solution u of (0.1) and write $v(t)=F_i(t, u^0(t))$, $w(t)=F_i(t, u^0(t))$, $w(t)=F_i(t, u(t))$. Since v is T-periodic,

$$(1/r)\int_s^{s+r} v(t)\, dt \to (1/T)\int_0^T v(t)\, dt \qquad \text{as} \quad r \to +\infty$$

uniformly on $s \in \mathbb{R}$.

On te other hand, $w(t)-v(t) \to 0$ as $t \to \infty$, since the partial derivatives $\partial F_i/\partial x_j$ are bounded in $\mathbb{R} \times K$ for all compact subsets K of $\mathbb{R}^n_+$. (Notice that these derivatives are periodic with respect to t.) From this,

$$(1/r)\int_s^{s+r} w(t)\, dt-(1/r)\int_s^{s+r} v(t)\, dt \to 0 \qquad \text{as} \quad s \to \infty$$

uniformly on $r>0$.

Assume now that $\int_0^T v(t)\, dt>0$. Then there exist s_0, $r_0>0$ such that $\int_s^{s+r} w(t)\, dt>0$ for $s>s_0$ and $r>r_0$. But

$$\int_s^{s+r} w(t)\, dt=\int_s^{s+r} (u_i'(t)/u_i(t))\, dt=\ln(u_i(s+r)/u_i(s))$$

and so, u_i is increasing in $[s_0, \infty)$. Contradiction, since $u_i(t) \to 0$ as $t \to \infty$. This contradiction ends the proof.

As a consequence of Theorem 1.5, we get

1.7. COROLLARY. *Let $G=(G_1, ..., G_n)$: $\mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function and suppose that there are positive constants m, $c_1, ..., c_n$ such that*

$$c_i \frac{\partial G_i}{\partial x_i}(x)+\sum_{j \in j_i} c_j \left|\frac{\partial G_j}{\partial x_i}(x)\right|+m \leqslant 0$$

for $1 \leqslant i \leqslant n$ and $x \geqslant 0$. Assume further that the system $u_i'=u_i G_i(u)$, $1 \leqslant i \leqslant n$, has a positive solution v, defined and bounded in $[0, \infty)$. Then, the system $u_i G_i(u)=0$, $1 \leqslant i \leqslant n$, has a solution u^0 in $\mathbb{R}^n_+$ such that $u(t) \to u^0$ as $t \to \infty$, for any positive solution u to the system $u_i'=u_i G_i(u)$, $1 \leqslant i \leqslant n$.

Remark, Let $H=(H_1, ..., H_n)$: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that: $H(t, 0) \equiv 0$; H is bounded in $\mathbb{R} \times K$ for all compact subsets K of $\mathbb{R}^n$; the partial derivatives $\partial H_i/\partial x_j$ are defined and continuous on $\mathbb{R} \times \mathbb{R}^n$, and

$$\frac{\partial H_i}{\partial x_j}(t, x)+\sum_{j \in J_i}\left|\frac{\partial H_j}{\partial x_i}(t, x)\right| \leqslant -m(t)$$

for $1 \leqslant i \leqslant n$. If (1.7) holds, then the trivial solution to system $x'=H(t, x)$ is globally exponentially stable. To show this, let $z(t)$ be a nontrivial solution of $x'=H(t, x)$; from the arguments in Theorems 1.1, 1.2, we get $\|z(t)\| \leqslant \|z(\tau)\| \exp(-\int_\tau^t m(s)\, ds)$ if $\tau \leqslant t$, τ, $t \in \operatorname{dom}(z)$; and hence $\operatorname{dom}(z)=(\alpha, \infty)$ for some $-\infty \leqslant \alpha \leqslant +\infty$, and $z(t) \to 0$ as $t \to \infty$.

Moreover, if (1.9) holds, then the trivial solution to the system $x'=H(t, x)$ is the only solution to this system bounded on $\mathbb{R}$. When H is a linear function of x, this result becomes the complement to Theorem 2 of [9].

2. COMPETITION SYSTEMS

We shall consider the system

$$u_i'=u_i\left[a_i(t)-\sum_{j=1}^n b_{ij}(t)\, u_j\right], \quad 1 \leqslant i \leqslant n, \tag{2.1}$$

where a_i, $b_{ij} \in C_+$ for $1 \leqslant i$, $j \leqslant n$. This system models the competition between n biological species.

2.1. PROPOSITION. *Suppose that there are positive constants ε, $M_1, ..., M_n$, such that $\varepsilon \leqslant M_1, ..., M_n$, $F_i(t, 0, ..., 0, M_i, 0, ..., 0) \leqslant 0$, and $F_i(t, M_1, ..., M_{i-1}, \varepsilon, M_{i+1}, ..., M_n) \geqslant 0$ for $1 \leqslant i \leqslant n$. Assume further that $F_i(t, x) \leqslant F_i(t, y)$ for $0 \leqslant x \leqslant y$).*

If u is a solution to (0.1) with $u(\tau)>0$ for some τ, then u is defined on $[\tau, \infty)$ and

$$\min\{u_1(\tau), ..., u_n(\tau), \varepsilon\} \leqslant u_i(t) \leqslant \max\{u_1(\tau), ..., u_n(\tau), M_i\} \quad \textit{for} \quad 1 \leqslant i \leqslant n \textit{ and } t \geqslant \tau. \tag{2.2}$$

Moreover, system (0.1) has a solution $u=(u_1, ..., u_n)$ such that

$$u_1, ..., u_n \in C_+. \tag{2.3}$$

Proof. Fix $1 \leqslant i \leqslant n$, and let N_i denotes the max in (2.2). Obviously, $u_i(\tau) \leqslant N_i$. Assume now that $u_i(t_2)>N_i$ for some $t_2>\tau$. Then there exists t_1, $\tau<t_1<t_2$, such that $u_i(t_1)>N_i$ and $u_i'(t_1)>0$. From this,

$0 \geqslant F_i(t_1, 0, ..., 0, M_i, 0, ..., 0) \geqslant F_i(t_1, u(t_1)) > 0$; and this contradiction proves the second inequality in (2.2) for $t \geqslant \tau$, $t \in \mathrm{dom}(u)$. In particular, u is defined on $[\tau, \infty)$, since (H_2) holds. The first inequality in (2.2) is proved by the same arguments.

Let us fix $x = (x_1, ..., x_n)$ in $\mathbb{R}^n$ such that $\varepsilon \leqslant x_i \leqslant M_i$ and let u^k be the solution to (0.1) given by $u^k(-k) = x$, for $k = 1, 2, ...$. From (2.2), $\varepsilon \leqslant u_i^k(t) \leqslant M_i$, for all components of u^k and $t > -k$. In particular, we can assume that $\{u^k(0)\}$ converges to $y \in \mathbb{R}^n$. Now, it is easy to show that the solution u of (0.1) given by $u(0) = y$, is a solution to (0.1)–(2.3) and so, the proof is complete.

2.2. THEOREM. *Assume*

$$1 > \sum_{j \in J_i} \sup(b_{ij}/a_i) \sup(a_j/b_{jj}), \qquad 1 \leqslant i \leqslant n \tag{2.4}$$

and let $M = (m_{ij})$ be the $n \times n$ matrix defined by $m_{ii} = 0$ and $m_{ij} = \sup(b_{ji}/b_{ii})$ for $i \neq j$. If M has no eigenvalues in $[1, \infty)$, then the problem (2.1)–(2.3) *has exactly one solution u^0 and $u(t) - u^0(t) \to 0$ as $t \to \infty$, for any positive solution u to* (2.1). *Moreover, u^0 is almost periodic (resp. T-periodic) if a_i, b_{ij} are almost periodic (resp. T-periodic).*

Proof. Let us define $M_i = \sup(a_i/b_{ii})$ and fix $\varepsilon > 0$ such that $\varepsilon \leqslant M_1, ..., M_n$ and

$$\varepsilon < M_i^{-1}\left[1 - \sum_{j \in J_i} M_j \sup(b_{ij}/a_i)\right], \qquad 1 \leqslant i \leqslant n.$$

If F_i is defined by (0.4) then the assumptions in Proposition 2.1 are satisfied and the proof will follow from Theorems 1.2 and 1.4, if we show that (1.1) holds.

To this end, let us fix $1 > \delta > 0$ such that the matrix $M_\delta = M + \delta$(identity) has no eigenvalues in $[1, \infty)$. From the Perron–Frobenius theory of positive matrices, we know that $M_\delta(c) = \lambda c$, for some λ in $(0, 1)$ and $c = \mathrm{col}(c_1, ..., c_n) > 0$. From here, $M(c) < (1 - \delta) c$, and hence

$$c_i b_{ii}(t) \geqslant m + \sum_{j \in J_i} c_j b_{ji}(t), \qquad 1 \leqslant i \leqslant n,$$

where $m := \delta \inf\{c_i b_{ii}(t)\colon t \in \mathbb{R},\ 1 \leqslant i \leqslant n\}$. This implies (1.1), and so the proof is complete.

2.3. COROLLARY. *If* (0.5) *holds then the assertions in Theorem* 2.2 *are true.*

Proof. It is clear that (0.5) implies (2.4). Define now the $n \times n$ matrix $P = (p_{ij})$ by $p_{ii} = 0$ and $p_{ij} = b_{ijM}/b_{jjL}$ for $i \neq j$. Then (0.5) implies $P(d) < d$, where $d := \mathrm{col}(a_{1L}, ..., a_{nL})$. In particular, spectral radius $M \leqslant$ radius spectral $P^* < 1$, where M is the matrix in Theorem 2.2 and P^* is the adjoint matrix to P. The proof follows now from Theorem 2.2.

The second assumption in Theorem 2.2 is satisfied if $n = 2, 3$ and $\det(I - M) > 0$, where I is the identity matrix. In particular, we get

2.4. COROLLARY. *Assume $n = 2$ and* $\inf(a_1/b_{12}) > \sup(a_2/b_{22})$, $\inf(a_2/b_{21}) > \sup(a_1/b_{11})$, *and* $\inf(b_{11}/b_{21}) > \sup(b_{12}/b_{22})$. *Then the assertions in Theorem* 2.2 *hold.*

Remarks. (a) Corollary 2.4 was proved in [3] in the periodic case. In the almost periodic case, this corollary improves the main results in [1].

(b) Corollary 2.3 generalizes the main results in [1, 2, 5, 6, 11].

As a consequence of Theorem 1.5 we get the following

2.5. COROLLARY. *Assume that a_i, b_{ij} are T-periodic for some $T > 0$. If $n = 2$ and* (0.6) *holds then system* (2.1) *has a nonnegative T-periodic solution u^0 such that, $u(t) - u^0(t) \to 0$ as $t \to \infty$, for any positive solution u of* (2.1).

The last result of this section is a consequence of Theorem 1.5 and Remark 1.6, which improves the main theorem in [11].

2.6. COROLLARY. *Assume that a_i, b_{ij} are T-periodic, and let U_i be the unique positive T-periodic solution to the logistic equation $x' = x[a_i(t) - b_{ii}(t)\, x]$. Suppose that*

$$\int_0^T \left[a_i(t) - \sum_{j \in J_i} b_{ij}(t)\, U_j(t)\right] dt > 0 \qquad 1 \leqslant i \leqslant n$$

and assume that there exist positive constants $c_1, ..., c_n$, such that

$$c_i b_{ii}(t) > \sum_{j \in J_i} c_j b_{ji}(t), \qquad 1 \leqslant i \leqslant n. \tag{2.5}$$

Then, system (2.1) *has a positive T-periodic solution u^0 and $u(t) - u^0(t) \to 0$ as $t \to \infty$, for any positive solution u to* (2.1).

Proof. Let u be a positive solution to (2.1). Then u is defined and bounded in $[\tau, \infty)$ if τ belongs to $\mathrm{dom}(u)$. On the other hand, (2.5) implies (0.2) and, by Theorem 1.5, there exists a nonnegative T-periodic solution u^0 to (2.1) such that $u(t) - u^0(t) \to 0$ as $t \to \infty$, for any positive solution u to (2.1).

Assume now that $u_i^0 = 0$ for some i and define F_i by (0.4). Then

$$F_i(t, u^0(t)) > a_i(t) - \sum_{j \in J_i} b_{ij}(t)\, U_j(t)$$

since $u_j^\circ < U_j$ if $u_j^\circ > 0$. See [11]. This contradicts Remark 1.6, and the proof is complete.

3. The Predator–Prey Model

In this section we consider the system

$$\begin{aligned} u' &= u[-a(t) - b(t)\, u + c(t)\, v] \\ v' &= v[d(t) - e(t)\, u - f(t)\, v], \end{aligned} \tag{3.1}$$

where $a, ..., f \in C_+$. The following proposition justifies assumption (3.3) in the main result of this section.

3.1. Proposition. *If* $\inf(f/d) \geqslant \sup(c/a)$ *then system* (3.1) *has no solution* (u, v) *such that*

$$u, v \in C_+. \tag{3.2}$$

Proof. To simplify our statements, let us define $B = b/a$, $C = c/a$, $E = e/d$, and $F = f/d$. Assume now that (u, v) is a solution to (3.1)–(3.2). It is not hard to prove that if $g: \mathbb{R} \to \mathbb{R}$ is bounded and differentiable, then there is a sequence (t_k) in $\mathbb{R}$ such that $g(t_k) \to g_L$ (resp. g_M) and $g'(t_k) \to 0$ as $k \to \infty$. Choose a sequence (t_k) such that $u(t_k) \to u_L$ and $u'(t_k) \to 0$ as $k \to \infty$. Then $-B(t)\, u(t_k) + C(t_k) \to 1$ and hence, $1 \leqslant -B_L u_L + C_M v_M$. Analogously, $1 \geqslant E_L u_L + F_L v_M$ and so, $0 \geqslant C_M - F_L \geqslant (C_M E_L + B_L F_L) > 0$. This contradiction ends the proof.

3.2. Theorem. *Suppose that* $a, ..., f$ *are* T*-periodic and*

$$\inf(c/a) > \sup(f/d) \tag{3.3}$$

$$\inf(b/e) > \sup(c/f). \tag{3.4}$$

Then, system (3.1) *has a positive* T*-periodic solution* (u_0, v_0) *such that*

$$(u(t) - u_0(t), v(t) - v_0(t)) \to (0, 0) \quad \text{as} \quad t \to \infty \tag{3.5}$$

for any positive solution (u, v) *of* (3.1).

Proof. Notice first that (3.3) is equivalent to $C_L > F_M$ and define $\alpha = (C_M - F_L)/B_L F_L$. If (u, v) is a positive solution to (3.1) with $u(0) \leqslant \alpha$ and $v(0) \leqslant 1/F_L$, we can prove that $u(t) \leqslant \alpha$ and $v(t) \leqslant 1/F_L$ for $t \geqslant 0$, $t \in \operatorname{dom}(u, v)$. From this (u, v) is defined and bounded in $[0, \infty)$. On the other hand, (3.4) implies (1.1) and by Theorem 1.5, there exists a nonnegative T-periodic solution (u_0, v_0) to (3.1), which satisfies (3.5).

Let (u, v) be a nonnegative T-periodic solution to (3.1) such that $uv = 0$. Then $u = 0$ and $v \in \{0, V\}$, where V is the unique positive T-periodic solution to the logistic equation $x' = x[d(t) - f(t)\, x]$. On the other hand, $1/F_M \leqslant V \leqslant 1/F_L$ and so, $cV - a \geqslant a_L[(C_L/F_M) - 1] > 0$. The proof follows from Remark 1.6 and the arguments in Corollary 2.6.

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