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On the Nonautonomous *N*-competing Species Problems

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Abstract We consider a nonautonomous system of ordinary differential equations which models competition among n species. Conditions are given under which there exists a unique solution with components bounded above and below by positive constants on $(-\infty, \infty)$ and which attracts all other solutions with positive components. In the case where this system is periodic or almost periodic in time, this unique solution is periodic or almost periodic. Our proofs make use of a combination of techniques from [6] and [8] and improve the results of these papers. An example is given to illustrate this improvement and another example shows that certain conditions which imply existence do not imply the conditions guaranteeing uniqueness and stability.

KEY WORDS: Lyapunov function, globally attracting solution

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We consider the system of differential equations

$$u_i'(t) = u_i(t)\left[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)u_j(t)\right] \tag{1}$$

for $1 \le i \le n$, *where the functions* $a_i(t), b_{ij}(t), 1 \le i, j \le n$, *are assumed to be nonnegative and continuous for* $t \in (-\infty, \infty)$. This system is commonly called the Volterra-Lotka system for n competing species. Many elementary differential equations texts treat the autonomous case when $n = 2$ and show that if $a_1 > b_{12}a_2/b_{22}$ and $a_2 > b_{21}a_1/b_{11}$, then there is an asymptotically stable equilibrium point with both coordinates positive such that its basin of attraction is the open first quadrant in the phase plane.

In [6], Gopalsamy considered the system (1) in which all the functions $a_i(t), b_{ij}(t)$ were all assumed to positive periodic functions, having the same period, and in [7] he considered the case where these functions were positive and almost periodic.

In order to describe the results in [6] and [7], as well as other related results we introduce some notation. Given a function $g(t)$ defined on $I\!R = (-\infty, \infty)$ we let

$$g_L = \inf_{I\!R} g(t), \quad g_M = \sup_{I\!R} g(t).$$

In [7] it was shown that if $a_i(t), b_{ij}(t)$ are all positive and periodic with the same period $T > 0$, then the two sets of conditions

$$a_{iL} > \sum_{\substack{j=1 \\ j\neq i}}^{N} b_{ijM} a_{jM} / b_{jjL}, \tag{2}$$

for $i = 1, ..., n$, and

$$b_{jjL} > \sum_{\substack{i=1 \\ i\neq j}}^{n} b_{ijM}, \tag{3}$$

for $j = 1, ..., n$, imply the existence of a solution $u_0 = col(u_{10}, u_{20}, ..., u_{n0})$ of (1) whose components are positive and T-periodic such that for any other solution $u = col(u_1, ..., u_n)$ of (1) with $u_i(t_0) > 0$, $1 \leq i \leq n$, for some $t_0 \in I\!R$ it follows that u is defined on $[t_0, \infty)$ and $u_{i0}(t) - u_i(t) \to 0$ as $t \to \infty$. In [7] it was shown that if the functions $a_i(t)$ and $b_{ij}(t)$ are positive and almost periodic, then the conditions (2) and (3) imply the existence of an almost periodic solution of (1) whose components are bounded below by positive constants on $(-\infty, \infty)$ and which attracts all other solutions u of (1) such that all components of $u(t_0)$ are positive at some time t_0.

In [2] it was shown that in the case when $n = 2$ and the functions $a_i(t)$ and $b_{ij}(t)$ are positive and T-periodic, the conditions (2) *alone* imply the existence of a T-periodic solution of (1), with positive components, which attracts all solutions starting in the first quadrant. This is what one might expect from what happens in the autonomous case. In [1] it was shown that if $n = 2$ and the functions $a_i(t), b_{ij}(t)$ are positive and almost periodic, and the conditions (2) *alone* hold, then there exist an almost periodic solution, with components bounded below by positive constants on $(-\infty, \infty)$ which attracts all solutions starting in the first quadrant.

In [8] it was shown that in the case of arbitrary n, if the functions $a_i(t)$ and $b_{ij}(t)$ are all positive and T-periodic, then the conditions (2) *alone* imply the same conclusions of Gopalsamy's main theorem in [6]. This involved a clever application of the Perron-Frobenius theorem concerning matrices with positive entries.

The purpose of this note is to give a result which pertains to a general nonautonomous Volterra-Lotka system and extends all of the above results. For example, it will follow that if the functions $a_i(t)$ and $b_{ij}(t)$ are T-periodic and the functions $b_{ii}(t)$ are positive for $i = 1, ..., n$, then the conditions

$$a_{iL} > \sum_{\substack{j=1 \\ j \neq i}}^{n} b_{ijM}(a_j/b_{jj})_M \quad i = 1, ..., n$$

will imply the conclusion of the main theorems of [6] and [7].

Throughout this note we make use of the well-known fact (see [1], [2], or [6]) that if $u = col(u_1, ..., u_n)$ is a solution of (1) with $u_i(t_0) > 0$ for $i = 1, ..., n$ and u is defined on $[t_0, t_1]$, then $u_i(t) > 0$ for $1 \leq i \leq n$ and $t_0 \leq t \leq t_1$.

In [6, p.69-70], Gopalsamy essentially established the following useful result.

Lemma 1. *Let $a_i(t)$ and $e_{ij}(t)$ be nonnegative and continuous for $t \in \mathbb{R}$ and $1 \leq i,\ j \leq n$. Let $V = col(V_1, ..., V_n)$ and $W = col(W_1, ..., W_n)$ be solutions of the system*

$$x_i'(t) = x_i(t)[a_i(t) - \sum_{j=1}^{n} e_{ij}(t) x_j(t)] \qquad i = 1, ..., n \tag{4}$$

on the interval $[t_0, t_1]$ such that the components of V and W are positive. If there exists a number $\delta > 0$ such that for all $t \in [t_0, t_1]$

$$e_{jj}(t) \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} e_{ij}(t) + \delta \qquad i = 1, ..., n, \tag{5}$$

and

$$\theta(t) = \sum_{i=1}^{n} | \, ln(V_i(t)/W_i(t)) \, |,$$

then

$$\theta(t_1) \leq \theta(t_0) - \delta \int_{t_0}^{t_1} \sum_{i=1}^{n} | \, V_i(s) - W_i(s) \, | \, ds \tag{6}$$

The lemma is proved by noting that for the absolutely continuous function $| ln(V_i(t)/W_i(t)) |$ one has for almost all t

$$\frac{d}{dt} | ln(V_i/W_i) | = \left(\frac{V_i'}{V_i} - \frac{W_i'}{W_i}\right) sgn(V_i - W_i)$$

$$\leq -e_{ii} | V_i - W_i | + \sum_{\substack{j=1 \\ j \neq i}}^{n} e_{ij} | V_j - W_j | .$$

Applying this to each summand in $\theta'(t)$ and rearranging terms one obtains

$$\theta'(t) \leq \sum_{j=1}^{n} \left(-e_{jj}(t) + \sum_{\substack{i=1 \\ i \neq j}}^{n} e_{ij}(t) \right) | V_j(t) - W_j(t) |$$

and (6) follows using (5).

In [6], Gopalsamy assumed that the functions $a_i(t)$ and $e_{ij}(t)$ were all periodic with period $\omega > 0$ and that for $j = 1, ..., n$

$$\min_{[0,\omega]} e_{jj}(t) > \sum_{\substack{i=1 \\ i \neq j}}^{n} \max_{[0,\omega]} e_{ij}(t).$$

We shall use a more general version of Lemma 1, which is motivated by the work of Tineo and Alvarez.

Corollary to Lemma 1. Assume that there exist constants $\alpha_1, ..., \alpha_n$, all positive such that for $t \in I\!R$ and $1 \leq j \leq n$

$$\alpha_j b_{jj}(t) > \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i b_{ij}(t) + \delta \tag{7}$$

where $\delta > 0$. Let $v = col(v_1, ..., v_n)$ and $w = col(w_1, ..., w_n)$ be two solutions of (1) with components which are positive for $t_0 \leq t \leq t_1$. If

$$\theta(t) = \sum_{i=1}^{n} | ln(v_i(t) \backslash w_i(t)) |, \tag{8}$$

then

$$\theta(t_1) \leq \theta(t_0) - \delta \int_{t_0}^{t_1} \sum_{i=1}^{n} \alpha_i^{-1} | v_i(s) - w_i(s) | ds. \tag{9}$$

Proof: If for $1 \le i \le n$ we set

$$V_i(t) = \alpha_i^{-1} v_i(t), \quad W_i(t) = \alpha_i^{-1} w_i(t)$$

and for $1 \le i,\ j \le n$ we set

$$e_{ij}(t) = \alpha_j b_{ij}(t),$$

then $V = col(V_1, ..., V_n)$ and $W = col(W_1, ..., W_n)$ are solutions of the system (4) and the conditions (5) hold. Since for $1 \le i \le n$, $\mid ln(v_i(t)/w_i(t)) \mid = \mid ln(V_i(t)/W_i(t)) \mid$, it follows from Lemma 1 that

$$\begin{aligned}\theta(t_1) &\le \theta(t_0) - \delta \int_{t_0}^{t_1} \mid V_i(s) - W_i(s) \mid ds \\ &= \theta(t_0) - \delta \int_{t_0}^{t_1} \alpha_i^{-1} \mid v_i(s) - w_i(s) \mid ds\end{aligned}$$

which proves the lemma.

Lemma 2. *Assume that the conditions (7) hold. If $v = col(v_1, ..., v_n)$ is a solution of (1) defined on $[t_0, \infty)$ such that the conditions*

$$0 < \varepsilon \le v_i(t) \le M \tag{10}$$

hold for $1 \le i \le n$ and $t \ge t_0$, where ε and M are constant, and $w = col(w_1, ..., w_n)$ is any other solution of (1) with $w_i(t_0) > 0$ for $1 \le i \le n$, then w is defined on $[t_0, \infty)$ and for $1 \le i \le n$

$$w_i(t) - v_i(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Proof: Let $\theta(t)$ be defined as in (8). If $w(t)$ is defined on $[t_0, t_1]$, then $\theta(t_1) \le \theta(t_0)$. From (9), it follows that there cannot exist $\bar{t} > t_0$ such that w is defined on $[t_0, \bar{t})$ and

$$\mid w_1(t) \mid + \cdots + \mid w_n(t) \mid \to \infty$$

as $t \to \bar{t}-$. From the standard theory of differential equations and the form of the system (1), w is defined on $[t_0, \infty)$.

From (9) we have that for $t > t_0$

$$\delta \sum_{i=1}^{n} \alpha_i^{-1} \int_{t_0}^{t} \mid v_i(s) - w_i(s) \mid ds \le \theta(t_0).$$

Hence, for $i = 1, ..., n$

$$\int_{t_0}^{\infty} | v_i(s) - w_i(s) | \, ds < \infty.$$

Let

$$m(t) = \max\{| v_i(t) - w_i(t) | \quad i = 1, ..., n\}.$$

Since $m(t) \leq | v_1(t) - w_1(t) | + \cdots + | v_n(t) - w_n(t) |$,

$$\int_{t_0}^{\infty} m(t) dt < \infty.$$

From this we infer the existence of a sequence of numbers $\{t_n\}_1^{\infty} \subset [t_0, \infty)$ such that $t_n \to \infty$ and $m(t_n) \to 0$ as $n \to \infty$. Since for $i = 1, ..., n$

$$\left| \frac{w_i(t_n)}{v_i(t_n)} - 1 \right| \leq \frac{1}{\varepsilon} \mid w_i(t_n - v_i(t_n) \mid \leq \frac{m(t_n)}{\varepsilon},$$

we see that $\theta(t_n) \to 0$ as $n \to \infty$. But $\theta(t)$ is nonincreasing on $[t_0, \infty)$, by (9), so $\theta(t) \to 0$ as $t \to \infty$. It follows that for $i = 1, ..., n$

$$\left| \frac{w_i(t)}{v_i(t)} - 1 \right| \to 0 \quad \text{as} \quad t \to \infty,$$

and since

$$| w_i(t) - v_i(t) | = | v_i(t) | \left| \frac{w_i(t)}{v_i(t)} - 1 \right| \leq M \left| \frac{w_i(t)}{v_i(t)} - 1 \right|,$$

this proves the lemma.

Lemma 3. *Assume that the conditions (7) hold. If $v = col(v_1, ..., v_n)$ and $w = col(w_1, ..., w_n)$ are two solutions of (1) such that v and w are defined for $-\infty < t \leq t_1$ and such that for $i = 1, ..., n$*

$$0 < \varepsilon \leq v_i(t) \leq M,$$

$$0 < \varepsilon \leq w_i(t) \leq M,$$

for $t \leq t_1$, where ε and M are constants, then $v = w$.

Proof: If for $t \leq t_1$ we define $\theta(t)$ as in (8), then there exists a constant c_0 such that $\theta(t) \leq c_0$ for $t \leq t_1$. Since, by (9), for $t_0 < t_1$,

$$\delta \int_{t_0}^{t_1} \sum_{i=1}^{n} \alpha_i^{-1} | v_i(s) - w_i(s) | \, ds \leq$$

$$\theta(t_0) - \theta(t_1) \leq \theta(t_0) \leq c_0,$$

it follows that for $i = 1, ..., n$

$$\int_{-\infty}^{t_1} \mid v_i(s) - w_i(s) \mid ds < \infty.$$

If $m(t)$ is defined as in the previous lemma, then

$$\int_{-\infty}^{t_1} m(s)ds < \infty,$$

so we infer the existence of a sequence $\{s_n\}_1^\infty$ in $(-\infty, t_1]$ such that

$$s_n \to -\infty \quad \text{and} \quad m(s_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Since for $i = 1, ..., n$

$$\left| \frac{w_i(s_n)}{v_i(s_n)} - 1 \right| = \frac{1}{v_i(s_n)} \mid w_i(s_n) - v_i(s_n) \mid \leq \frac{m(s_n)}{\varepsilon},$$

it follows that $\theta(s_n) \to 0$ as $n \to \infty$. But θ is nonincreasing so for $n \geq 1$

$$0 \leq \theta(t_1) \leq \theta(s_n).$$

Hence $\theta(t_1) = 0$ so $w(t_1) = v(t_1)$ and $w = v$, which proves the lemma.

Lemma 4. *Assume that in addition to being continuous and nonnegative the functions $a_i(t)$ $i = 1, ..., n$, are bounded above and below by positive constants, the functions $b_{ii}(t)$, $i = 1, ..., n$ are bounded below by positive constants, and the functions $b_{ij}(t)$, $1 \leq i$, $i \leq n$ are bounded on $I\!R$. If for $i = 1, ..., n$*

$$a_{iL} > \sum_{\substack{j=1 \\ j \neq i}}^{n} b_{ijM} \left(\frac{a_j}{b_{jj}} \right)_M, \tag{11}$$

then there exist positive constants $\alpha_1, ..., \alpha_n$ and $\delta > 0$ such that the conditions (7) hold for all t.

Proof: We use a modification of the reasoning used by Tineo and Alvarez [8] to show that in the periodic case Gopalsamy's conditions imply the conditions (7).

Assuming that the conditions (11) hold, we have that for $i = 1, ..., n$

$$b_{iiL} \left(\frac{a_i}{b_{ii}} \right)_M = \inf_{t \in I\!R} \left(b_{ii}(t) \left(\frac{a_i}{b_i} \right)_M \right)$$

$$\geq \inf_{t \in I\!R} a_i(t) = a_{iL} > \sum_{\substack{j=1 \\ j \neq i}}^{n} b_{ijM} \left(\frac{a_j}{b_{jj}} \right)_M$$

If for $1 \leq i$, $j \leq n$ we set

$$m_{ij} = \begin{cases} b_{ijM}/b_{iiL} & i \neq j \\ 0 & i = j \end{cases}$$

and we set

$$\gamma_j = \left(\frac{a_j}{b_{jj}}\right)_M \quad j = 1, \ldots, n,$$

then we have the conditions

$$\sum_{j=1}^{n} m_{ij}\gamma_j < \gamma_i, \quad i = 1, \ldots, n.$$

For $1 \leq i,\ j \leq n$ let p_{ij} be a number such that

$$p_{ij} > m_{ij} \tag{12}$$

and such that

$$\sum_{j=1}^{n} p_{ij}\gamma_j < \gamma_i, \quad i = 1, \ldots, n. \tag{13}$$

If P is the $n \times n$ matrix such that the entry in the i-th row and j-th column is p_{ij} and P^T denotes the transpose of P, then P^T is a strictly positive matrix. By the Perron-Frobenius theorem [4], P^T has an eigenvector all of whose components are positive so there exist numbers $z_i > 0, 1 \leq i \leq n$, and $\lambda > 0$ such that

$$\sum_{i=1}^{n} p_{ij} z_i = \lambda z_j, \quad j = 1, \ldots, n. \tag{14}$$

From (13) and (14), it follows that

$$\lambda \sum_{j=1}^{n} \gamma_j z_j = \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}\gamma_j z_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_{ij}\gamma_j \right) z_i < \sum_{i=1}^{n} \gamma_i z_i.$$

Hence $\lambda < 1$, so from (14),

$$\sum_{i=1}^{n} p_{ij} z_i < z_j, \quad j = 1, \ldots, n.$$

Therefore, from (12), we have

$$\sum_{i=1}^{n} m_{ij} z_i < z_j, \quad j = 1, \ldots, n$$

or

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{b_{ijM}}{b_{iiL}} z_i < z_j$$

for $j = 1, \ldots, n$. Finally, setting

$$\alpha_j = \frac{z_j}{b_{jjL}}, \quad j = 1, \ldots, n,$$

we have that

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} b_{ijM} \alpha_i < b_{jjL} \alpha_j$$

for $j = 1, \ldots, n$. If we choose $\delta > 0$ so that

$$\alpha_j b_{jjL} - \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i b_{ijM} > \delta$$

for $j = 1, \ldots, n$, then it is clear that the conditions (7) hold for all t and the lemma is proved.

Lemma 5. *Assume that, in addition to being nonnegative and continuous, the functions $a_i(t)$ and $b_{ij}(t)$, $1 \leq i,\ j \leq n$, satisfy the following conditions:*

$$b_{ii}(t) > 0, \quad t \in \mathbb{R}, \qquad 1 \leq i \leq n,$$

$$(a_i/b_{ii})_M < \infty, \qquad 1 \leq i \leq n,$$

and for $1 \leq i \leq n$

$$\inf_{t \in \mathbb{R}} \left[\frac{a_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{n} b_{ij}(t)(a_j/b_{jj})_M}{b_{ii}(t)} \right] > 0. \tag{15}$$

Let $\varepsilon > 0$ be so small that

$$\varepsilon < (a_i/b_{ii})_M, \quad 1 \leq i \leq n, \tag{16}$$

and for $1 \leq i \leq n$ *and* $t \in \mathbb{R}$

$$a_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{n} b_{ij}(t)(a_j/b_{jj})_M > \varepsilon b_{ii}(t). \tag{17}$$

If $t_0 \in \mathbb{R}$ *and* $u = col(u_1, ..., u_n)$ *is a solution of (1) such that*

$$\varepsilon < u_i(t_0) < (a_i/b_{ii})_M \tag{18}$$

for $i = 1, ..., n,$ *then*

$$\varepsilon \leq u_i(t) \leq (a_i/b_{ii})_M \tag{19}$$

for $t \geq t_0$ and $i = 1, ..., n$.

Proof: If $s > 1$, then for $1 \leq i \leq n$, and $t \in \mathbb{R}$

$$\begin{aligned}
&a_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{n} s b_{ij}(t)(a_j/b_{jj})_M \\
&> s\varepsilon b_{ii}(t) - (s-1)a_i(t) \\
&\geq b_{ii}(t)[s\varepsilon - (s-1)(a_i/b_{ii})_M] \\
&\equiv b_{ii}(t)\varepsilon_1(s).
\end{aligned}$$

Clearly, we can choose $s > 1$ with $s - 1$ so small that

$$0 < \varepsilon_1(s) < s(a_i/b_{ii})_M$$

for $1 \leq i \leq n$.

Let $t_0 \in \mathbb{R}$ and let $u = col(u_1, ..., u_n)$ be a solution of (1) such that the inequalities (18) hold. For $s > 1$ and $s - 1$ small we have

$$\varepsilon_1(s) < u_i(t_0) < s(a_i/b_{ii})_M \tag{20}$$

for $1 \leq i \leq n$. We claim that

$$\varepsilon_1(s) < u_i(t) < s(a_i/b_{ii})_M \tag{21}$$

for $t > t_0$ and $i = 1, ..., n$. Indeed, in the contrary case, there would exist an integer k between 1 and n and a number $t_1 > t_0$ such that the inequalities (21) hold for $t_0 \le t < t_1$ and either:

(a) $u_k(t_1) = s(a_k/b_{kk})_M$

or

(b) $u_k(t_1) = \varepsilon_1(s)$.

If case (a) holds, then $u_k'(t_1) \ge 0$. However, according to (1) we would have

$$u_k'(t_1) \le u_k(t_1)[a_k(t_1) - sb_{kk}(t_1)(a_k/b_{kk})_M] < 0,$$

which is a contradiction, so case (a) cannot hold and $u_j(t_1) < s(a_j/b_{jj})_M$ for $j = 1, ..., n$. In case (b) we would have $u_k'(t_1) \le 0$, but, from (1)

$$u_k'(t_1) \ge u_k(t_1)\left[a_k(t_1) - s\sum_{\substack{j=1\\ j\ne k}}^{n} b_{kj}(t_1)(a_j/b_{jj})_M - b_{kk}(t_1)\varepsilon_1(s)\right] > 0.$$

This contradiction shows that inequalities (21) hold for $t > t_0$.

Since $\varepsilon_1(s) \to \varepsilon$ as $s \to 1-$ for $i = 1, ..., n$, it follows that the inequalities (19) must hold for all $t > t_0$ and the lemma is proved.

Theorem 1. *If the conditions of Lemma 5 hold, then there exists a solution $u_0 = col(u_{10}, ..., u_{n0})$ of (1) defined on $(-\infty, \infty)$ such that there exist constants M and ε such that for $i = 1, ..., n$ and $t \in \mathbb{R}$*

$$0 < \varepsilon \le u_{i0}(t) \le M. \tag{22}$$

If the conditions Lemma 4 hold, then there exists a unique solution u_0 of (1) whose components are bounded above and below by positive constants on $(-\infty, \infty)$, and if $u = col(u_1, ..., u_n)$ is any other solution with $u_i(t_0) > 0$ for $1 \le i \le n$ and some $t_0 \in \mathbb{R}$, then for $1 \le i \le n$

$$u_{i0}(t) - u_i(t) \to 0 \quad as \quad t \to \infty. \tag{23}$$

If the hypotheses of Lemma 4 hold and all of the functions $a_i(t)$, $b_{ij}(t)$, $1 \leq i, j \leq n$ are periodic with period T, then u_0 is T-periodic. If all these functions are almost periodic, then u_0 is also almost periodic.

Proof: Assume that the hypotheses of Lemma 5 hold. For each positive integer $m = 1, 2, \ldots$ let $u_m = col(u_{m1}, \ldots, u_{mn})$ be a solution of the system (1) such that $\varepsilon < u_{mi}(-m) < (a_i/b_{ii})_M$, where $\varepsilon > 0$ has the same meaning as in Lemma 5. It follows from Lemma 5 that $\varepsilon \leq u_{mi}(t) \leq (a_i/b_{ii})_M$ for $-m \leq t < \infty$. Using (1), it follows for any compact subinterval I of $[-m, \infty)$ the sequences of derivatives $\{u'_{mi}(t)\}_1^\infty$, $i = 1, \ldots, n$, are also uniformly bounded on I. Using the same diagonal argument as that used in Lemma 1 of [1], which is based on Ascoli's lemma, we infer the existence of a subsequence $\{u_{mk}\}_1^\infty$ of $\{u_m\}_1^\infty$ and a solution u_0 of (1) such that the components of $\{u_{mk}\}_1^\infty$ converge to the corresponding components of u_0 uniformly on compact subintervals of $(-\infty, \infty)$. If $u_0 = col(u_{10}, \ldots, u_{n0})$ then $\varepsilon \leq u_{i0}(t) \leq (a_i/b_{ii})_M$ for $i = 1, \ldots, n$ and $t \in \mathbb{R}$. This proves the first part of Theorem 1.

Obviously the conditions of Lemma 4 imply those of Lemma 5, so the conditions of Lemma 4 imply the existence of $u_0 = col(u_{10}, \ldots, u_{n0})$ such that u_0 is a solution of (1) and the conditions (22) hold on $(-\infty, \infty)$. Since the hypothesis of Lemma 4 imply the existence of positive numbers $\alpha_1, \ldots, \alpha_n$ and $\delta > 0$ such that the conditions (7) hold on $(-\infty, \infty)$, it follows from Lemma 3, that if the conditions of Lemma 4 hold, then the system (1) has a unique solution defined on $(-\infty, \infty)$ whose components are bounded above and below by positive constants on $(-\infty, \infty)$. From Lemma 4 and Lemma 2 we see that if $u = col(u_1, \ldots, u_n)$ is any solution of (1) whose components are positive at $t_0 \in \mathbb{R}$, then (23) holds for $1 \leq i \leq n$.

If the hypotheses of Lemma 4 hold, all the functions $a_i(t)$ and $b_{ij}(t)$ are periodic with period $T > 0$, and u_0 is as above, then both $u_0(t)$ and $u_0(t + T)$ are solutions with components bounded above and below by positive constants on $(-\infty, \infty)$, hence $u_0(t) \equiv u_0(t + T)$.

Suppose that the conditions of Lemma 4 hold and all of the functions $a_i(t)$, $b_{ij}(t)$ are almost periodic. If $\{\tau_m\}_1^\infty$ is a real numbers and

$$\lim_{m \to \infty} a_i(t + \tau_m) = a_i^*(t),$$

$$\lim_{m \to \infty} b_{ij}(t + \tau_m) = b_{ij}^*(t),$$

where the convergence is uniformly with respect to $t \in (-\infty, \infty)$, then the conditions of Lemma 4 will be satisfied for the system

$$u_i'(t) = u_i(t)[a_i^*(t) - \sum_{j=1}^{n} b_{ij}^*(t)u_j(t)] \qquad i = 1, \ldots, n, \tag{24}$$

so it will have a unique solution defined on $(-\infty, \infty)$ whose components are bounded above and below by positive constants on $(-\infty, \infty)$. It follows from a result due to Amerio [3,5] (or see [1] for the idea of a self-contained proof) that the unique solution u_0 of (1) satisfying (22) on $(-\infty, \infty)$ is almost periodic.

We conclude with some examples.

I. Consider the system

$$u_1'(t) = u_1(t)[3 + \cos t - (2 + \cos t)u_1(t) - 2u_2(t)]$$

$$u_2'(t) = u_2(t)[4 - \cos t - u_1(t) - (5 - \cos t)u_2(t)].$$

Writing this system in the form

$$u_i'(t) = [a_i(t) - b_{i1}(t)u_1(t) - b_{i2}(t)u_2(t)] \tag{25}$$

$i = 1, 2$, we have that

$$a_{1L} = 2, \quad a_{1M} = 4, \quad a_{2L} = 3, \quad a_{2M} = 5$$

$$b_{11L} = 1, \quad b_{12M} = 2, \quad b_{21M} = 1, \quad b_{22L} = 4.$$

Therefore

$$a_{1L} < b_{12M}a_{2M}/b_{22L}, \quad a_{2L} < b_{21M}a_{1M}/b_{11L}$$

so Gopalsamy's conditions (1) are not satisfied. However,

$$(a_1/b_{11})_M = \max_t((3 + \cos t)/(2 + \cos t)) = 2,$$

$$(a_2/b_{22})_M = \max_t((4 - \cos t)/(5 - \cos t)) = 5/6$$

so

$$a_{1L} > b_{12M}(a_2/b_{22})_M, \quad a_{2L} > b_{21M}(a_1/b_{11})_M.$$

Therefore, from Theorem 1, we can conclude that the above system has a unique 2π-periodic solution whose components are strictly positive for all t which attracts all other solutions which start out in the positive first quadrant.

II. One might hope that the conditions of Lemma 5 imply the conditions (7). However, the following example shows that this is not true.

Consider the system

$$u_1'(t) = u_1(t)[3 + \cos t - (3 + \cos t)u_1(t) - (2 + \cos t)u_2(t)]$$

$$u_2'(t) = u_2(t)[4 - \cos t - (3 - \cos t)u_1(t) - (4 - \cos t)u_2(t)].$$

Writing this in the form (25) we have

$$a_1(t)/b_{11}(t) \equiv 1 \equiv a_2(t)/b_{22}(t)$$

so

$$\frac{a_1(t) - b_{12}(t)(a_2/b_{22})_M}{b_{11}(t)} = \frac{1}{b_{11}(t)} \geq \frac{1}{2},$$

$$\frac{a_2(t) - b_{21}(t)(a_1/b_{11})_M}{b_{22}(t)} = \frac{1}{b_{22}(t)} \geq \frac{1}{3},$$

so the hypotheses of Lemma 5 are satisfied. However, if there were constants $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\alpha_1 b_{11}(t) > \alpha_2 b_{21}(t) + \delta$$

$$\alpha_2 b_{22}(t) > \alpha_1 b_{12}(t) + \delta$$

for all t, where $\delta > 0$ is constant, then

$$\frac{b_{11}(t)}{b_{21}(t)} > \frac{\alpha_2}{\alpha_1} > \frac{b_{12}(t)}{b_{22}(t)}$$

for all t. Consequently it would follow that

$$\min \frac{b_{11}(t)}{b_{21}(t)} > \max \frac{b_{12}(t)}{b_{22}(t)}.$$

But

$$\min \frac{b_{11}(t)}{b_{21}(t)} = \min \left(\frac{3 + \cos t}{3 - \cos t}\right) = \frac{1}{2}$$

and

$$\max \frac{b_{12}(t)}{b_{22}(t)} = \max \left(\frac{2 + \cos t}{4 - \cos t}\right) = 1,$$

and we would have a contradiction.

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