Chapter One

Circuits

(... introduction here ...)

1.1 CIRCUIT BASICS

Objects may possess a property known as electric charge. By convention, an electron has one negative charge (-1) and a proton has one positive charge ($+1$). In the presence of an electric field, the electromagnetic force acts on electrically charged objects, accelerating them in the direction of the force.

We denote by Q the net positive charge at a given point of an electrical circuit. Thus $Q > 0$ means there are more positive charged particles than the negatives ones, whereas $Q < 0$ means the opposite. Through a point of (or an imaginary plane with a fixed direction that is perpendicular to) a circuit, Q may change (or flow) with time and we denote it by $Q(t)$ with t being the time variable. By definition, its rate of change (or the *flux* through the plane in the direction of the imaginary plane) is defined as the *current*

$$I = \frac{dQ}{dt}. \tag{1.1}$$

In reality though, it is usually the negative charged electrons that actually move. In such cases, the net 'positive' charge Q would move opposite to the actual direction of the free electrons. Electric charges can move in one direction at one time and the opposite direction at another. In circuit analysis, however, there is no need to keep track the actual direction in real time. Instead, we only need to assign the direction of the current through a point *arbitrarily* once and for all (or to be the same direction as the imaginary plane through the point and perpendicular to the circuit). Notice that this assignment in turn mutually and equivalently designates the *reference* direction of the movement of the net positive charge Q. In this way, $I > 0$ means the net positive charge Q flows in the reference direction while $I < 0$ means it flows in the opposite direction. Again, in most cases, free electrons actually move in the oppositive direction of I when $I > 0$ and the same direction of I when $I < 0$. To emphasize, we state the following rule.

Rule of Current Direction Assignment:

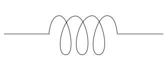
> The reference direction of the current through a point or an electrical device of a circuit is assigned arbitrarily and kept the same throughout the subsequent analysis of the circuit.

This does not mean that all assignments are equally convenient. In practice, some assignments make the analysis easier than others. Fundamentally, however, all choices are equally valid and equivalent.

Between two points in an electric field, we envision the existence of a potential difference, and the field to be the result of the gradient of the potential. More precise, the electric field is the negative gradient of the potential. The potential difference, measured in voltage, or volt, can be thought as "electric pressure" that push electric charges down the potential gradient. However, we do not need to keep track of the actual potential, but instead use the reference direction of the current to define the reference voltage. More precisely, at a given point of a circuit, the side into which the reference direction of the current points is considered to be the high end of the potential whereas the other side from which the current leaves is the low end of the potential. Hence, $I > 0$ at the point if and only if $V \geq 0$ with V denoting the voltage across the point.

The case with $V \equiv 0$ is special — it represents a point on an ideal conductor without resistance. Here below, however, we will consider various elementary cases for which the mutual dependence between V on I, which we refer to as the IV**-characteristic curve** or IV**-characteristic**, is not this trivial kind.

By **device** in this chapter, we mean an inductor, or a capacitor, or resistor, which are uniquely defined by their IV-characteristics.



Inductor
$V = L\frac{dI}{dt}$

An **inductor** is an electrical device that stores energy in a magnetic field, typically constructed as a coil of conducting materials. The voltage across an inductor with inductance L with the current through it is described by the equation

$$V = L\frac{dI}{dt}. \tag{1.2}$$

It is the IV-characteristic of inductors. In one interpretation, a positive change of the current results a voltage drop across the inductor. Alternatively, a potential difference either accelerates (when $V > 0$) or decelerates (when $V < 0$) the positive charge Q through the inductor. It is this IV-characteristic that allows us to model processes and devices as an inductor even though they are not made of coiled wires.

Capacitor
$V = \frac{1}{C}Q$

A **capacitor** is another electrical device that stores energy in the electric field. Unlike an inductor, it does so between a thin gap of two conducting plates on which equal but opposite electric charges are placed. Its capacitance (C) is a measure of the conducting material as to how much of a voltage between the plates for a given charge

$$V = CQ. \tag{1.3}$$

Since the rate of change in charge is the current $I = \frac{dQ}{dt}$, we also have

$$\frac{dV}{dt} = C\frac{dQ}{dt} = CI,$$

which can be considered to define the IV-characteristic of the capacitor.

A word of clarification is needed for capacitors. For a given direction of the current I, the side that I goes into a capacitor is the "positive" side of the capacitor, and the voltage is fixed by the relation $V = Q/C$ with Q being the "net positive" charge. It means the following, let Q^+ be the net positive charge on the positive side of the capacitor and Q_+ be net the positive charge on the opposite side of the capacitor, then $Q = Q^+ - Q_+$. In this way, the current "flows" from the "net positive" charged side of the capacitor plate to the "net negative" charged side of the plate. In reality, however no actual charges flow across the plates which are insulated from each other. Instead, positive charges are repelled from the "net negative" charged side or equivalently negative charges are attracted to the "net negative" charged side, giving the effect that it is as if the current "flows" across the plates.

Resistor
$V = RI$ or
$I = GV$

Variable Resistor
$g(V, I) = 0$

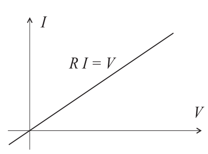
A **resistor** is an electrical device that is used to create a simpler voltage-to-current relationship. It has two broad categories: linear and variable resistors. A linear resistor has a constant voltage to current ratio, the resistance R, by the Ohm's law

$$V = RI. \tag{1.4}$$

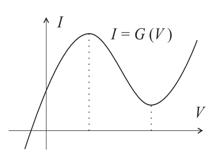
Alternatively, a resistor is also referred to as a conductor, with the conductance

$$G = \frac{1}{R} \text{ and } I = GV. \tag{1.5}$$

For a variable resistor, however, the relationship between V and I is nonlinear in general. There are three cases. For one case, V is a nonlinear function of I, $V = f(I)$, often referred to as a variable (or nonlinear) resistor. For



linear characteristics



nonlinear characteristics

another case, I is a nonlinear function of V, $I = f(V)$, referred to as a variable (or nonlinear) conductor. For the third category, neither V nor I can be expressed as a function of the other, but nonetheless constrained by an equation, $F(V, I) = 0$. The corresponding curve defined by such an equation is still called the IV-characteristic the nonlinear resistor (or nonlinear resistive or passive network as referred to in literature). It can also be thought as the level curve of $F = 0$. Notice that the first two cases can also be expressed as $F(V, I) = \pm[I - f(V)]$ or $F(V, I) = \pm[V - f(I)]$ with the exception that either I or V can be solved explicitly from $F(V, I) = 0$.

A linear resistor is characterized by its resistance R, which is the constant, positive slope of the IV-characteristic curve. Heat dissipates through the resistor resulting the potential drop in voltage. The situation can be different on a nonlinear resistor. Take the nonlinear IV-characteristic curve, $I =$

Quantity	Unit	Conversion
Elementary Charge		1 elementary charge = the charge of a single electron (−1) or proton (+1)
Charge Q	coulomb (C)	1 C = 6.24×10^{18} elementary charges
Current $I = \frac{dQ(t)}{dt}$	ampere (A)	1 A = 1 C/s (s = second)
Potential Difference V or E	volt (V)	1 V = 1 Newton-meter per C
Inductance L	henry (H)	1 H = 1 Vs/A
Capacitance C	farad (F)	1 F = 1 C/V
Resistance R	ohm Ω	1 Ω = 1 V/A
Conductance $g = 1/R$	siemens (S)	1 S = 1 A/V

$f(V)$, shown as an example. It has two critical points that divide the V-axis into three intervals in which $I = f(V)$ is monotone. In the voltage range defined by the left and right intervals, the slope of f is positive. As result, the device behaves like a resistor as expected, dissipating energy. However, in the voltage range of the mid interval, the slope of f is negative. It does not behave like a resistor as we think it would — a faster current results in a smaller voltage drop. That is, instead of dissipating energy, energy is drown to the device, and hence the circuit that it is in. In electronics, nonlinear resistors must be constructed to be able to draw and store energy, usually from batteries.

Useful Facts about Elementary Devices:

- For each device, we only need to know one of the two quantities, the current or the voltage, because the other quantity is automatically defined by the device's IV-characteristics (1.1–1.5).
- Once the direction of a current is assigned, the corresponding voltage is automatically fixed by the device relations (1.1–1.5).
- The change of voltage and current through an inductor and capacitor are gradual, transitory in time.
- In contrast, the relationship between the voltage and current across a resistor is instantaneous, at least for an idealized resistor or conductor.
- Unit and conversion for these basic devices are given in the company table. As an example for interpretation, consider the unit of voltage. According to its unit, one can visualize that one volt is the amount of energy that is required to move one coulomb of charge by a force of one Newton for one meter.

Exercises 1.1

1.
2.

1.2 KIRCHHOFF'S LAWS

In a circuit, the voltage and current across a given device change with time. As functions of time, they satisfy a set of differential equations, which in turn are considered as a mathematical model of the circuit. All equations are based on two laws of conservation, one for the conservation of energy — *Kirchhoff's voltage law*, and the other for the conservation of mass — *Kirchhoff's current law.*

Kirchhoff's Voltage Law

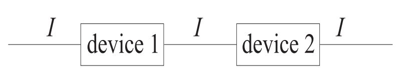
> The directed sum of the electrical potential differences around a circuit loop is zero.

Kirchhoff's Current Law

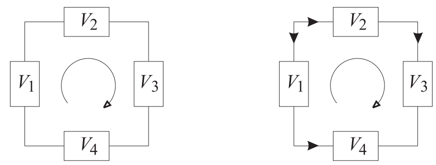
> At any point of an electrical circuit where charge density is not changing in time, the sum of currents flowing towards the point is equal to the sum of currents flowing away from the point.

Remark.

1. Two devices are connected in **series** if the wire connecting them does not branch out in between and connect a third device. As one immediate corollary to Kirchhoff's Current Law, the current through all devices in a series is the same. This negates the need to assign a current to each device in a series.



2. Although individual devices in a circuit loop can be assigned arbitrary current directions, and their voltages are thus fixed thereafter by the formulas (1.1–1.5), the "directed sum" stipulation in Kirchhoff's Voltage Law *does not* mean to simply add up these arbitrarily defined voltages. It means the following. We first pick an orientation for the loop, for example, the counterclockwise direction. We then either add or subtract the voltage V of a device in the "directed sum". It is to add if V's current direction is the same as the orientation of the loop. It is to subtract if the current direction is against the orientation of the loop. Two examples are illustrated below. For the left loop, the currents through the V_i devices with voltage V_i are implicitly taken to be the same counterclockwise orientation of the loop. In contrast, each device from the right loop is given an individual reference direction of its current.

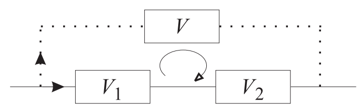


Voltage Law: $V_1 + V_2 + V_3 + V_4 = 0$ $\qquad -V_1 + V_2 + V_3 - V_4 = 0$

3. **Voltage Additivity in Series.** As an immediate corollary to the Voltage Law, the voltages over devises in series are additive. That is, if $V_1, V_2, \ldots, V_n$ are voltage drops across devices $1, 2 \ldots, n$, then the voltage drop V over all devices is

$$V = V_1 + V_2 + \cdots + V_n.$$

The diagram below gives an illustration of the required argument.



Voltage Law: $-V_1 - V_2 + V = 0$ or $V = V_1 + V_2$

We wire a resistor with infinite resistance $R = \infty$ in parallel with the serial devices as shown. The new wire draws zero current from the existing circuit. By Kirchhoff's Voltage Law (and the previous remark), we see that the voltage across the infinite resistor is the numerical sum of the existing serial voltages.

Because of the second last remark, we can fix the direction of a device in a circuit as follows for convenience and definitiveness.

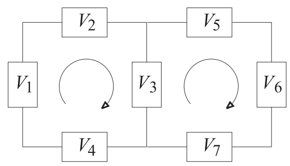
Current Direction Convention:

Unless specified otherwise, the following convention is adopted:

- If a circuit consists of only one loop, then all device currents are counterclockwise oriented.
- If a circuit consists of a network of parallel loops, all device currents in the left most loop are counterclockwise oriented. Device currents of subsequent addition to the existing loop that forms the new loop are counterclockwise oriented as well.

Remark. Useful tips to note: the current of a device that is in two adjacent loops is counterclockwise oriented with respect to the left loop but clockwise oriented with respect to the right loop. Its voltage takes different signs in the directed sums of the loops.

Example 1.2.1 According to the Current Direction Convention, the circuit below has three equations by Kirchhoff's Laws as shown.



Voltage Law:

left loop: $V_1 + V_2 + V_3 + V_4 = 0$

right loop: $-V_3 + V_5 + V_6 + V_7 = 0$

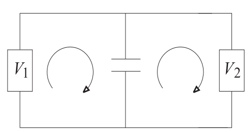
Current Law:

top and bottom nodes: $I_2 = I_3 + I_5$

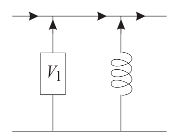
Notice that both the top and bottom node points give rise to the same equation of currents by the Current Law because $I_2 = I_1 = I_4$ and $I_5 = I_6 = I_7$.

Exercises 1.2

1. In each of the circuits below, label appropriate currents, use Kirchhoff's Current Law and Voltage Law to write the equations governing the voltages and the currents through the devices.

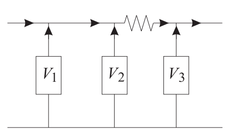


(a)

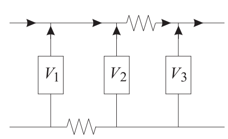


(b)

2. In each of the circuits below, label appropriate currents, use Kirchhoff's Current Law and Voltage Law to write the equations governing the voltages and the currents through the devices.



(a)



(b)

1.3 CIRCUIT MODELS

Circuit analysis is to study how devices currents and voltages change in a circuit. The main reason why a circuit can be completely understood is because of the following principles.

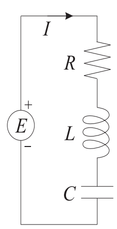
- Each device has only one independent variable, either the current or the voltage, and the other is determined by the device relations (1.1–1.5).
- If a circuit has only one loop, then all the devices are in series, and we only need to know the current variable. Since the circuit loop gives rise to one equation from the Voltage Law, we should solve one unknown from one equation.

- If a circuit has two parallel loops, such as the circuit diagram of Example 1.2.1, then there are three serial branches—left, middle, and right branches—each is determined by its current variable, or one device voltage variable in general. However, the node points give rise to only one independent equation from the Current Law, and each loop gives rise to one equation from the Voltage Law. Together we have three equations for three variables, and it is solvable in principle.
- The same argument above applies to multiple parallel loops.

To illustrate these principles, we consider the following examples for which only the equations as mathematical models of circuits are derived. The equations inevitably contain derivatives of variables. Such equations are not *algebraic* rather than *differential* equations. Methods to solve the differential equations are considered in later sections.

Linear Circuits

Example 1.3.1 *RLC* **Circuit in Series.** Consider the *RLC* circuit as shown. We only need to know the current variable I. From the Voltage Law we have



$$V_R + V_L + V_C - E = 0,$$

with the subscript of the voltage V_i corresponding to the devices (R for the resistor, L for the inductor, and C for the capacitor). Since batteries have a designated positive terminal, its corresponding current is fixed as well. Thus, in this case it is opposite the reference current direction chosen, and hence the negative sign of E in the directed sum equation. In principle, this equation together with the device relations (1.1–1.5),

$$V_R = RI,\ V_L = L\frac{dI}{dt},\ V_C = \frac{1}{C}Q,\ \text{ and } I = \frac{dQ}{dt}$$

completely determines the circuit.

However, circuit analysis does not stop here. We usually proceed to reduce the number of equations as much as possible before we solve them by appropriate methods for differential equations.

To illustrate the point, we express all voltages in terms of the current I by the device relations and substitute them into the voltage equation. More specifically, we first have

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E. \tag{1.6}$$

Differentiating the equation above in t yields,

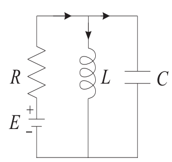
$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}. \tag{1.7}$$

Alternatively, (1.7) can be directly expressed in term of Q as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E. \tag{1.8}$$

Either of the last two equations models the circuit mathematically. They are now ready to be solved by appropriate methods for differential equations which is a different issue from the model building completed above.

We note that the equation above is a **second order**, **linear**, **ordinary differential equation**(ODE). The term "order" refers to the highest derivative taken in the equation. The term "linear" refers to the fact that the expression on the left side is is a linear combination of the unknown I in derivatives: $d^2I/dt^2, dI/dt, I$. The term "ordinary" is given to equations for which the unknown is a function of only one variable. If an equation has an unknown which is a function of more than one variables and contains partial derivatives of the unknown, then it is called **partial differential equation**. We will discuss numerical methods to solve differential equations in later sections.



Example 1.3.2 *RLC* **Circuit in Parallel.** Consider the RLC circuit in parallel as shown. By the Current Law, we have

$$I_R = I_L + I_C,$$

for both points. By the Voltage Law, we have

$$V_R + V_L - E = 0 \text{ and } V_C - V_L = 0$$

for the left and right loop respectively.

Like the previous example, we can derive a second order, linear ODE of the inductor current I_L for the circuit, which is left as an exercise. Here we derive alternatively a system of two equations of two variables, V_C and I_L.

From the capacitor relation $V_C = Q/C$, the current definition $dQ/dt = I_C$, and the node point current equation $I_C = I_R - I_L$, we have

$$\frac{dV_C}{dt} = \frac{1}{C} I_C = \frac{1}{C}(I_R - I_L). \tag{1.9}$$

Replacing I_R by the resistor relation $I_R = V_R/R$ and $V_R = E - V_L = E - V_C$ from both loop voltage equations gives rise to the first equation

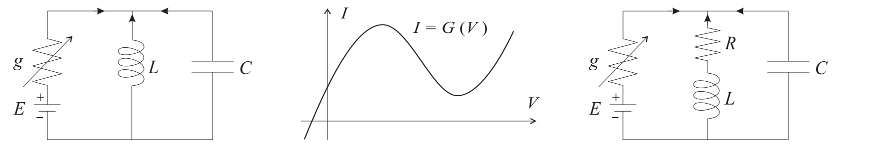
$$\frac{dV_C}{dt} = \frac{1}{C}\left(\frac{1}{R}(E - V_C) - I_L\right).$$

From $V_C = V_L$ of the right loop and the inductor relation $L dI_L/dt = V_L$, we have the second equation

$$L\frac{dI_L}{dt} = V_C.$$

Together, we have obtained a system model of the circuit

$$\begin{cases} C\dfrac{dV_C}{dt} = \dfrac{1}{R}(E - V_C) - I_L \\ L\dfrac{dI_L}{dt} = V_C. \end{cases} \tag{1.10}$$



(a) (b) (c)

Figure 1.1 (a) van der Pol circuit. (b) IV-characteristics. (c) FitzHugh-Nagumo circuit.

The main difference between Eqs.(1.7, 1.8) and Eq.(1.10) is that the former are second order equations whereas the latter is a system of two first order equations. Geometric and numerical methods work the best for systems of first order equations. For this reason, all higher order ODEs are converted to systems of first order ODEs before geometric and numerical methods are applied.

Nonlinear Circuits

Circuits above contain only linear resistors. We now consider circuits with nonlinear resistors (or conductors).

Neuron Circuit Current Convention.

> Unless specified otherwise, all reference directions of electrical currents through the cell membrane of a neuron are from inside to outside. All circuit diagrams for neurons are placed in such a way that the bottom end of the circuit corresponds to the interior of the cells and the top end to the exterior of the cell.

Example 1.3.3 van der Pol Circuit. The circuit is the same as the RLC parallel circuit except for a nonlinear resistor, both are shown in Fig.1.1. For the nonlinear conductance g, we use the following polynomial form for the nonlinear resistive characteristics

$$I = F(V) = V(aV^2 + bV + c)) + d, \tag{1.11}$$

where $a > 0, b, c, d$ are parameters.

Applying Kirchhoff's Voltage and Current Laws, we have

$$V_R - E - V_L = 0,\ V_L - V_C = 0,\ I_C + I_R + I_L = 0. \tag{1.12}$$

From the capacitor and current relation $V_C = Q/C, dQ/dt = I_C$, and the equations above, we have

$$\frac{dV_C}{dt} = \frac{1}{C} I_C = \frac{1}{C}(-I_R - I_L). \tag{1.13}$$

Replacing I_R by the nonlinear resistor relation $I_R = F(V_R)$ and R_R by $V_R = E + V_L = E + V_C$ from Eq.(1.12) gives rise to the first equation

$$C\frac{dV_C}{dt} = -F(E + V_C) - I_L.$$

From $V_C = V_L$ of Eq.(1.12) and the inductor relation $LdI_L/dt = V_L$, we have the second equation

$$L\frac{dI_L}{dt} = V_C.$$

Finally, we have the model system of equations

$$\begin{cases} C\dfrac{dV_C}{dt} = -F(E + V_C) - I_L \\ L\dfrac{dI_L}{dt} = V_C. \end{cases} \tag{1.14}$$

Example 1.3.4 FitzHugh-Nagumo Circuit. The circuit is the same as the van der Pol circuit except that a linear resistor is in series with the inductor as shown in Fig.1.1. The nonlinear conductance g has the same form as van der Pol's circuit.

It is left as an exercise to derive the following system of equations for the circuit, referred to as **FitzHugh-Nagumo equations**.

$$\begin{cases} C\dfrac{dV_C}{dt} = -F(E + V_C) - I_L \\ L\dfrac{dI_L}{dt} = V_C - RI_L. \end{cases} \tag{1.15}$$

Notice that the only difference from the van der Pol equations is on the right hand side of the second equation. As result, the van der Pol equations is a special case of the FitzHugh-Nagumo equations with $R = 0$. For this reason, the latter reference will be used often. Another reason that will become apparent later is because FitzHugh-Nagumo equations are better suited for a class of neuron circuits.

Forced Circuits

The circuits considered so far are "closed" networks of devices. In applications, they may be a part of a larger circuit, having their own incoming and outgoing ports through which connections to the larger circuit are made. The current going into the compartment is thus considered as an external forcing. The circuit including the two ports is therefore considered as a forced circuit of the closed one without the interphase ports.

Example 1.3.5 Forced FitzHugh-Nagumo Circuit. The circuit is the same as the FitzHugh-Nagumo circuit except for the additional I_{in} and I_{out} branches as shown in Fig.1.2. It is left as an exercise to show that

$$I_{in} = -I_{out} = -(I_g + I_R + I_C).$$

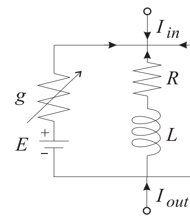


Figure 1.2 Forced FitzHugh-Nagumo circuit.

It is also left as an exercise to derive the **forced FitzHugh-Nagumo equations** for the circuit,

$$\begin{cases} C\dfrac{dV_C}{dt} = -F(E+V_C) - I_L - I_{in} \\ L\dfrac{dI_L}{dt} = V_C - RI_L. \end{cases} \tag{1.16}$$

We will also interchangeably refer to it as FitzHugh-Nagumo equations because the closed circuit version corresponds to the case when $I_{in} = 0$.

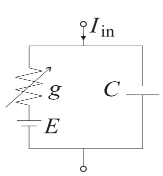
Exercises 1.3

1. Show that the equation for the RLC circuit in parallel from Example 1.3.2 can also be written as the following second order, linear, ODE,

$$RCL\frac{d^2I_L}{dt^2} + L\frac{dI_L}{dt} + RI_L = E.$$

2. Use the current relation $dQ/dt = I$ to transform the equation Eq.(1.6) for the RLC serial circuit into a system of two first order ODEs.
3. Derive the FitzHugh-Nagumo equations (1.15).
4. Show that for the forced FitzHugh-Nagumo circuit from Fig.1.2, $I_{in} = -I_{out}$.
5. Derive an ODE for the RC circuit shown. The nonlinear conductance g is given by the IV-characteristics $I = F(V)$.
6. Derive the forced FitzHugh-Hagumo equations (1.16).
7. Show the forced FitzHugh-Nagumo equations (1.16) can be transformed into the following second order, nonlinear ODE,

$$L\frac{d^2I_L}{dt^2} + R\frac{dI_L}{dt} + \frac{1}{C}I_L + \frac{1}{C}F(E + L\frac{dI_L}{dt} + RI_L) = -\frac{1}{C}I_{in}.$$

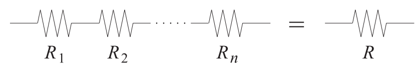


Problem 5

1.4 EQUIVALENT CIRCUITS

A circuit is a network of devices. If we are only interested in a current, I_{in}, that goes into the circuit at one point, called the **input node**, another current I_{out}, that come out at another point, called the **output node**, and the voltage drop between these points, $V_{in,out}$, then whatever happen in between is not important. Loosely speaking, two circuits are **equivalent** if they are interchangeable between the input and output node points without altering the quantities $I_{in}, I_{out}, V_{in,out}$.

Example 1.4.1 A set of n linear resistors in series is equivalent to a resistor whose resistance is equal to the sum of individual resistors resistance $R_k, k = 1, 2, \ldots, n$.



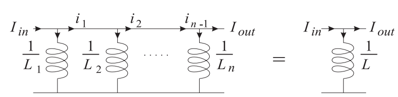
A verification follows from the property of Voltage Additivity in Series from the remarks after Kirchhoff' Voltage Law in Sec.1.2 that the voltage drop across devices in series is the sum of individual devices voltage. In particular, the input and output currents I_{in}, I_{out} are the same, I. Since the voltage drop across all resistors is

$$V = V_1 + V_2 + \cdots + V_n = R_1 I + R_2 I + \cdots R_n I = (R_1 + R_2 + \cdots + R_n)I,$$

it is equivalent to one resistor of the resistance

$$R = R_1 + R_2 + \cdots + R_n.$$

Example 1.4.2 A set of n linear inductors in parallel is equivalent to an inductor whose resistance in reciprocal is equal to the sum of individual inductors inductance in reciprocal: $\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$.



By the Current Direction Convention and Kirchhoff's Voltage Law, we have

$$-V_1 = V_2 = V_3 = \cdots = V_n = V$$

for the equivalence conditions for voltages, with the first equation from the left most loop, the second equation from the second left loop, etc. The last equation is the condition for an equivalent inductor. Let $i_k, k = 1, 2, \ldots, n-1$ be the currents running on the top edges of the $n-1$ loops. Then, by Kirchhoff's Current Law, we have

$$I_{in} + I_1 = i_1, \; i_k = I_{k+1} + i_{k+1}, \; i_{n-1} = I_n + I_{out},$$

with the first equation from the top left node, the $(k+1)$th node from left, and the nth or the last node. Use the formulas recursively, we have

$$I_{in} = -I_1 + i_1 = -I_1 + I_2 + i_2 = \cdots = -I_1 + I_2 + \cdots + I_n + I_{out}.$$

Since for the equivalent circuit we have

$$I_{in} = I + I_{out},$$

we must have

$$I = -I_1 + I_2 + \cdots + I_n$$

as the equivalence condition for current. Differentiating this equation in time, and using the device relations $dI_k/dt = V_k/L_k$, we have

$$\frac{1}{L}V = -\frac{1}{L_1}V_1 + \frac{1}{L_2}V_2 + \cdots + \frac{1}{L_n}V_n.$$

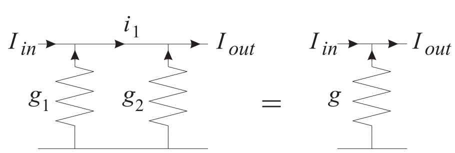
Since $V = V_k$ for $k \geq 2$ and $V = -V_1$ from the equivalence conditions for voltages, we have

$$\frac{1}{L}V = \left[\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}\right]V.$$

Canceling V from both sides yields the equivalence condition for the inductance.

Equivalent Nonlinear Conductors. For linear devices, the equivalent characteristics can be expressed in simple algebraic relations. For nonlinear devices, it is usually not as simple. However, the principle reason why equivalence exists at all is the same. It all depends on Kirchhoff's Voltage and Current Laws.

Consider two nonlinear resistors in parallel as shown with IV-characteristics $g_1(V, I) = 0$ and $g_2(V, I) = 0$ respectively.



By Kirchhoff's Voltage Law, $V_1 = V_2$. By the voltage equivalence condition,

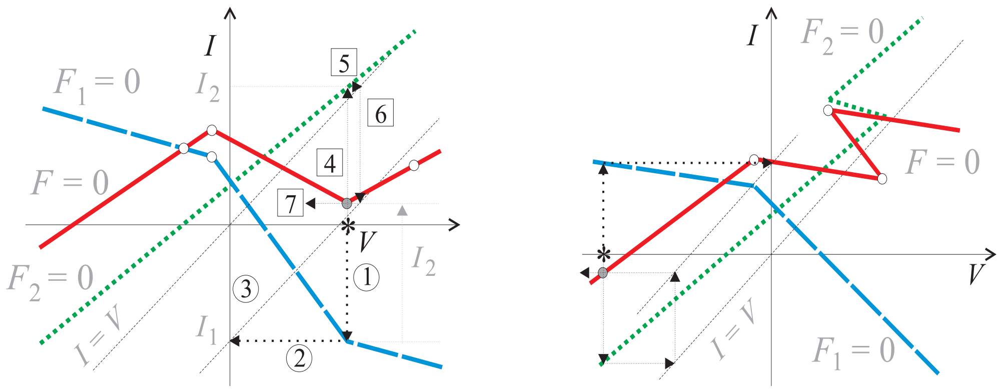
$$V_1 = V_2 = V. \quad (1.17)$$

By Kirchhoff's Current Law, $I_{in} + I_1 = i_0$ and $i_0 + I_2 = I_{out}$. Eliminating i_0, $I_{in} + I_1 + I_2 = I_{out}$. By the current equivalence condition, $I_{in} + I = I_{out}$. Together, they imply

$$I = I_1 + I_2. \quad (1.18)$$

Equations.(1.17, 1.18) mean the following. For a given voltage V, if I_1 and I_2 are corresponding currents on the resistors in parallel, $g_k(V, I_k) = 0, k = 1, 2$, then the current I on the equivalent nonlinear resistor, $g(V, I) = 0$, must be the sum of I_1 and I_2. Geometrically, all we have to do is for every pair of points $(V, I_1), (V, I_2)$ from the constituent IV-curves, plot the point $(V, I_1 + I_2)$ on the same vertical line $V = V$ in the plane to get the equivalent IV-characteristic curve.

There are two ways to do this.



Example 1.4.3 Example 1.4.4

Figure 1.3 Graphical Method for equivalent resistors of resistors in parallel.

Methods of Deriving Equivalent IV-Characteristics in Parallel

Graphical Method. This method will produce the equivalent IV-characteristics for two resistors in series or in parallel as long as the constituent IV-curves are given. Illustration is given for the parallel equivalent resistor case, and the serial case is left as an exercise.

Example 1.4.3 Consider two nonlinear resistors in parallel with their nonlinear IV-characteristics $F_1(V, I) = 0, F_2(V, I) = 0$ as shown. For clarity, piecewise linear curves are used for both curves. The problem is to place the point $(V, I_1 + I_2)$ graphically on each vertical line $V = V$ which intersects the constituent IV-curves, i.e. $F_1(V, I_1) = 0, F_2(V, I_2) = 0$.

Here are the steps to follow.

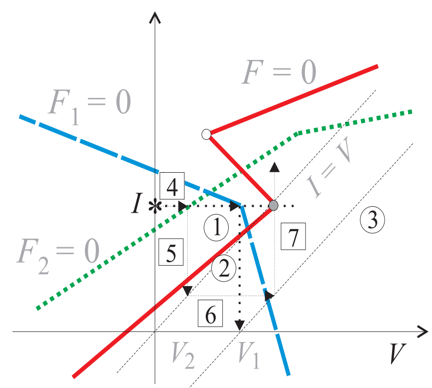
- *Step 1.* Plot both constituent IV-curves in one IV-plane. Draw the diagonal line $I = V$ in the plane (the line through the origin with slope $m = 1$).
- *Step 2.* Locate all **critical points** of the constituent IV-curves. A point is critical if it is not a smooth point of the curve, or it has a horizontal tangent line, or a vertical tangent line. Any point at which the curve changes its monotonicity in either I or V direction is a critical point.
- *Step 3.* To fix notation, let $(V, I_1), (V, I_2)$ be a pair of points from the F_1 and F_2 characteristic curves respectively, i.e., $F_k(V, I_k) = 0$, so that one of the two is a critical point and both are on the same vertical line $V = V$. The method now proceeds in sub-steps that are numbered as illustrated. For example, start at the critical value V labelled as $*$. Choose one of the two constituent IV-curves we want to start. Our illustration takes F_1 as an example. Draw a vertical line from $*$ to the graph F_1. If there are multiple intersections, do one intersection at a time. In sub-step number 2, draw a horizontal line from the intersection to the I-axis. From the intersection, draw a line parallel to the diagonal

to complete sub-step 3. Sub-step 4 starts from the critical point $*$ again, and do the same for the other IV-curve, $F_2 = 0$ for the illustration. Sub-step 7 draws a horizontal line to intersect the vertical line $V = V$. The intersection point is $(V, I_1 + I_2)$.

- *Step 5.* Repeat the same procedures for all critical points, and two far end "anchoring" points chosen arbitrarily.
- *Step 6.* Since the sum of two lines is another line, (here is the algebra: $(b_1 + m_1 x) + (b_2 + m_2 x) = (b_1 + b_2) + (m_1 + m_2)x$), we now connect the "dots" which are produced branch by branch. The resulting curve is the *exact* equivalent IV-characteristics. If the constituent IV-curves are not lines, we only need to "smooth" the dots to get an *approximate* of the equivalent curve, which is often sufficient for qualitative purposes.

Example 1.4.4 Figure 1.3 shows another example of the equivalent IV-characteristics for two nonlinear resistors in parallel. Again, the equivalent curve is constructed by the graphical method. A noncritical point is used to illustrate the steps which are not numbered.

Example 1.4.5 The graphical method can be extended to nonlinear resistors in series. Let the IV-characteristics of nonlinear resistors be $F_1(V, I) = 0, F_2(V, I) = 0$ respectively. Then it is similar to argue by Kirchhoff's Current Law that for each fixed current I, if V_1, V_2 are the voltages on the two constituent characteristics, i.e., $F_k(V_k, I) = 0, k = 1, 2$, then the equivalent resistor's characteristics must be $V = V_1 + V_2$ for the same current level I. To locate the equivalent voltage $V = V_1 + V_2$, we only need to interchange vertical and horizontal constructing lines in the graphical method illustrated above for nonlinear resistors in parallel. The figure gives an illustration of the graphical method.



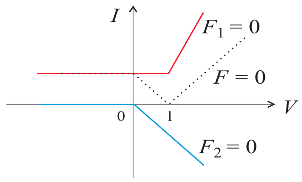
Analytical Method.

- If each I_k can be written as a function of V, $I_k = f_k(V)$, then the equivalent characteristics is simply

$$I = f_1(V) + f_2(V) = f(V).$$

- Suppose I_1 can be written piece by piece as a set of n functions of V, $I_1 = f_k(V), 1 \le k \le n$, and I_2 can be written piece by piece as a set of m functions of V, $I_2 = g_j(V), 1 \le j \le m$. Then the equivalent characteristics can be pieced together by no more than mn functions of V, $I = f_k(V) + g_j(V)$ for $k = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The single functions case above is just a special case.

Example 1.4.6 Consider two nonlinear resistors in parallel whose IV-characteristics g_1, g_2 are as shown. For g_1, I_1 can be written piecewise as



$$I_1 = g_1(V) = \begin{cases} 1, & \text{for } V \le 1 \\ 2V - 1, & \text{for } V \ge 1. \end{cases} \tag{1.19}$$

For g_2, I_2 can be written piecewise as

$$I_2 = g_2(V) = \begin{cases} 0, & \text{for } V \le 0 \\ -V, & \text{for } V \ge 0. \end{cases} \tag{1.20}$$

The equivalent characteristic g can be put together with no more than $2\times 2 = 4$ pieces. In fact, in the interval $(-\infty, 0]$, the piece of f_2 overlaps the left component of f_1, eliminating one unnecessary combination with the right component. The equivalent characteristics in that interval is $I = I_1 + I_2 = 1{+}0 = 1$. For the remaining interval $[1, \infty)$, the right piece of f_2 overlaps with both pieces of f_1 in range. Hence we break it up into two pieces, one over the interval $[0, 1]$ and the other $[1, \infty)$. In the mid interval, $I = I_1 + I_2 = 1 - V$, and in the right interval, $I = I_1 + I_2 = 2V - 1 - V = V - 1$. To summarize, we have

$$I = g(V) = \begin{cases} 1, & \text{for } V \le 0 \\ 1 - V, & \text{for } 0 < V \le 1 \\ V - 1, & \text{for } 1 \le V. \end{cases} \tag{1.21}$$

The equivalent IV-curve consists of the dotted lines.

We note that this method can be implemented computationally.

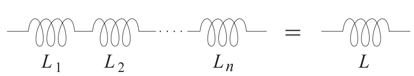
In general, there is no method to write down the equivalent IV-characteristic, $F(V, I) = 0$, for a parallel or serial network of resistors whose IV-characteristics are implicitly defined. However, this can be done for some special cases. For example, if one resistor's IV-curve is given implicitly as $F_1(V, I) = 0$ and another resistor's IV-curve is given explicitly with I as a function of V: $I = f_2(V)$. Then, the equivalent IV-characteristic $F(V, I) = 0$ for the two resistors in parallel is

$$F(V, I) = F_1(V, I - f_2(V)) = 0,$$

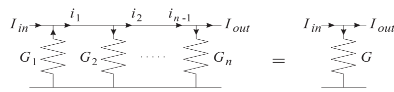
where $I = I_1 + I_2$ with $I_2 = f_2(V)$ and $F_1(V, I_1) = 0$. This is left as an exercise to the reader.

Exercises 1.4

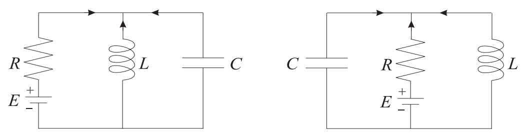
1. Show that a set of n inductors in series with inductance $L_k, k = 1, 2, \ldots, n$ is equivalent to an inductor whose inductance is the sum of individual inductors inductance.



2. Show that a set of n resistors in parallel with conductance $G_k = 1/R_k, k = 1, 2, \ldots, n$ is equivalent to resistor whose conductance is the sum of individual resistors conductance.

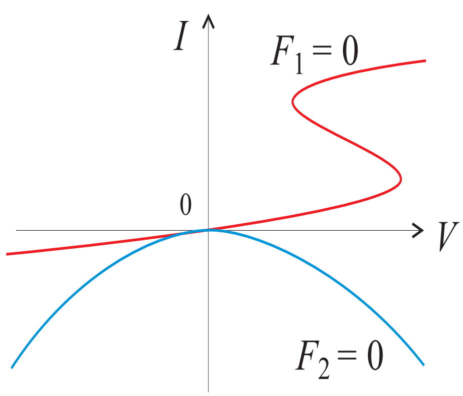


3. Show that the following circuits are equivalent in the sense that both have the set of equations. That is, the order by which circuit loops are laid out is not important. (When electrons moves in a circuit, it follows physical laws of energy conservations, not arbitrary order of circuit paths we lay out on a planar diagram.)

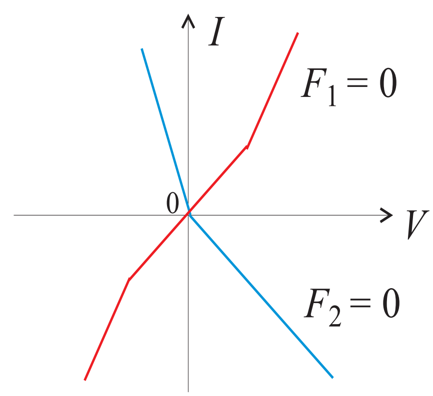


For the following problems, label the sub-steps in constructing one critical point of your choice.

4. Use the graphical method to sketch the equivalent IV-characteristics of Example 1.4.6.
5. Use the graphical method to sketch the equivalent IV-characteristic curve of two resistors in parallel whose IV-characteristic curves are as shown.
6. Use the graphical method for serial nonlinear resistors to sketch the equivalent IV-characteristic curve of two resistors in series whose IV-characteristic curves are as shown.



Problem 5



Problem 6

7. For problem 5 above, if $F_1(V, I) = V - I(I^2 + 4I + 5)$ and $F_2(V, I) = V^2 + I$, find the function form $F(V, I)$ so that $F(V, I) = 0$ defines the IV-characteristic of the equivalent resistor with F_1, F_2 in parallel.