Stable and Unstable Manifolds II

This notes is about stable and unstable manifolds for nonhyperbolic fixed points of diffeomorphisms.

Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $J = Df(\bar{q})$, and denote

$$\sigma^s = \sigma(J) \cap \{|z| < 1\}, \sigma^c = \sigma(J) \cap \{|z| = 1\}, \text{ and } \sigma^u = \sigma(J) \cap \{|z| > 1\}$$

the set of stable eigenvalues, center eigenvalues, unstable eigenvalues, respectively, of the linearizatoin $Df(\bar{q})$. Let

$$\sigma^{cs} = \sigma^s \cup \sigma^c, \text{ and } \sigma^{cu} = \sigma^c \cup \sigma^u.$$

Definition 1. *Let $\bar{q}$ be a nonsingular fixed point of a differentiable mapping f in $\mathbb{R}^d$ and α be any constant satisfying*

$$\max\{|\sigma^s|\} < \alpha < 1.$$

The stable manifold of the fixed point $\bar{q}$ for f is

$$W^{\mathrm{s}} = \{p : \{\alpha^{-n}[f^n(p) - \bar{q}]\}_{n=0}^{\infty} \textit{ is a bounded sequence.}\}.$$

Theorem 1 (Stable Manifold Theorem)**.** *Let $\bar{q}$ be a nonsingular fixed point of a diffeomorphism f in $\mathbb{R}^d$ with splitting $\mathbb{R}^d \cong \mathbb{E}^s \oplus \mathbb{E}^{cu}$. Then for sufficiently small $\delta > 0$, $\|f - Df(\bar{q})\|_1 < \delta$ implies the definition of W^{s} is independent of any two different choices in the constant α. Also W^{s} is the graph of a C^1 function $\phi_{cu} : \mathbb{E}^s \to \mathbb{E}^{cu}$*

$$W^{\mathrm{s}} = \mathrm{graph}(\phi_{cu}),$$

and the tangent space of W^{s} at the fixed point is the stable eigenspace

$$\mathbb{T}_{\bar{q}} W^{\mathrm{s}} = \mathbb{E}^s.$$

Moreover, f is a contraction mapping on W^{s}. Furthermore, if f is C^k, $k \geq 1$, and all its derivatives $D^j f$, $1 \leq j \leq k$, are bounded, then ϕ_{cu} is also C^k with bounded derivatives.

The proof is an application of the Uniform Contraction Principle. The main idea is to construct the stable-manifold function ϕ_{cu} as part of a fixed point of a uniform contraction map. We will break it up into a few lemmas.

Before doing so, we first translate $\bar{q}$ to the origin and choose a coordinate system (x, y) for the splitting in which $Df(\bar{q}) \cong \mathrm{diag}(A_s, A_{cu})$. By the Variation of Parameters Formula Theorem, for sufficiently small $\|f - Df(\bar{q})\|_1$, the map $(\bar{x}, \bar{y}) = f(x, y)$ is equivalent to

$$\left\{ \begin{array}{l} \bar{x} = A_s x + h_s(x, y) \\ y = A_{cu}^{-1} \bar{y} + h_{cu}(\bar{x}, \bar{y}), \end{array} \right. \tag{1}$$

and for any orbit, $p_n = (x_n, y_n) = f^n(p_0),\ n \geq 0$, the following Variation of Parameters Formula (VPF) holds

$$\begin{cases} x_n = A_s^n x_0 + \sum_{i=1}^n A_s^{n-i} h_s(p_{i-1}) \\ y_n = A_{cu}^{n-m} y_m + \sum_{i=n+1}^m A_{cu}^{n+1-i} h_{cu}(p_i). \end{cases} \tag{2}$$

Also, by the VPF theorem, the functions $h_s,\ h_{cu}$ are all C^1 satisfying

$$h_s(0) = 0,\ Dh_s(0) = 0,\ h_{cu}(0) = 0,\ Dh_{cu}(0) = 0, \tag{3}$$

and they are globally Lipschitz and the Lipschitz constant can be taken to be

$$L = \|Dh\|_0 \to 0 \quad \text{as} \quad \|f - Df(\bar{q})\|_1 \to 0. \tag{4}$$

We will repeatedly use this formula for geometric sequences

$$a + ar + ar^2 + \cdots + ar^{n-1} = \tfrac{a(1-r^n)}{1-r}, \text{ for } r \neq 1$$

and its differentiation formulas in r.

Lemma 1. *For the parameter α satisfying $\max\{|\sigma^s|\} < \alpha < 1$, let*

$$S_\alpha := \{\gamma = \{p_n\}_{n=0}^\infty : p_n \in \mathbb{R}^d, \sup\{\alpha^{-n}\|p_n\| : n \geq 0\} < \infty\} \tag{5}$$

for an adapted norm $\|\cdot\|$ for $\mathbb{R}^d$ and for any $\gamma \in S_\alpha$ let

$$\|\gamma\|_\alpha = \sup\{\alpha^{-n}\|p_n\| : n \geq 0\}.$$

For any $\gamma \in S_\alpha$, $\gamma = \{p_n\}_{n=0}^\infty$, let $\bar{\gamma} = T(\gamma)$ be defined by the equations below

$$\begin{cases} \bar{x}_n = A_s^n x_0 + \sum_{i=1}^n A_s^{n-i} h_s(p_{i-1}) \\ \bar{y}_n = \sum_{i=n+1}^\infty A_{cu}^{n+1-i} h_{cu}(p_i). \end{cases} \tag{6}$$

Then $\bar{\gamma} \in S_\alpha$ with

$$\|\bar{\gamma}\|_\alpha \leq \|x_0\| + \tfrac{L\|\gamma\|_\alpha}{\alpha-\nu} + \tfrac{L\alpha\|\gamma\|_\alpha}{1-\alpha\beta} \tag{7}$$

where ν, β are fixed parameters satisfying

$$\max\{|\sigma^s|\} < \nu < \alpha < 1 < \beta < 1/\alpha. \tag{8}$$

More importantly, $p \in W^{\mathrm{s}}$ if and only if the orbit $\gamma_p = \{f^n(p)\}_{n=0}^\infty$ is a fixed point of T with

$$p = (x_0, y_0) = \left(x_0, \textstyle\sum_{i=1}^\infty A_{cu}^{1-i} h_{cu}(p_i)\right). \tag{9}$$

Proof. For α, β, ν satisfying (8), choose an adapted norm so that the following inequalities hold

$$\|A_s\| < \nu < \alpha < 1, \ \text{ and } \ \|A_{cu}^{-1}\| < \beta < 1/\alpha. \tag{10}$$

For $\gamma \in S_\alpha,\ \bar\gamma = T(\gamma)$, since $\|h_s(p)\| = \|h_s(p) - h_s(0)\| \le L\|p\|$ we have

$$\begin{array}{rl} \|\bar x_n\| & \le \|A_s^n\|\|x_0\| + \sum_{i=1}^n \|A_s^{n-i} h_s(p_{i-1})\| \\ & \le \nu^n\|x_0\| + \sum_{i=1}^n \nu^{n-i} L\alpha^{i-1}\|\gamma\|_\alpha \\ & = \nu^n\|x_0\| + L\|\gamma\|_\alpha \frac{\alpha^n-\nu^n}{\alpha-\nu} \\ & \le (\|x_0\| + \frac{L\|\gamma\|_\alpha}{\alpha-\nu})\alpha^n. \end{array} \tag{11}$$

Similarly, because $\|A_{cu}^{n+1-i}\| \le \beta^{i-n-1}, i \ge n+1$, $\alpha\beta < 1$, and $\|h_{cu}(p)\| = \|h_{cu}(p) - h_{cu}(0)\| \le L\|p\|$,

$$\begin{array}{rl} \|\bar y_n\| & \le \sum_{i=n+1}^\infty \|A_{cu}^{n+1-i} h_{cu}(p_i)\| \\ & \le \sum_{i=n+1}^\infty \beta^{i-n-1} L\alpha^i \|\gamma\|_\alpha \\ & = \beta^{-n-1} L\|\gamma\|_\alpha \frac{(\alpha\beta)^{n+1}}{1-\alpha\beta} \\ & = \frac{L\alpha\|\gamma\|_\alpha}{1-\alpha\beta}\alpha^n. \end{array} \tag{12}$$

Hence, the estimate (7) for $\bar\gamma$ holds, and T maps S_α into itself.

Now for any $p = p_0 = (x_0, y_0) \in W^{\mathrm{s}}$, because $\gamma = \{p_n = (x_n, y_n) = f^n(p)\}_{n=0}^\infty \in S_\alpha$, $\|p_n\| \le \|\gamma\|_\alpha \alpha^n$. Because $\|A_{cu}^{n-m} y_m\| \le \beta^{m-n}\|\gamma\|_\alpha \alpha^m$ and $\alpha\beta < 1$, the first term of the y_n-equation of the VPF (2) goes to zero as $m \to \infty$. By the same estimate as for (12), the partial sum of the y_n-equation of the VPF (2) converges uniformly. Therefore, by taking $m \to \infty$, we see that the stable-manifold orbit γ is a fixed point of T.

Conversely, if a sequence $\gamma = \{p_n = (x_n, y_n)\} \in S_\alpha$ is a fixed point of T, satisfying

$$\left\{ \begin{array}{l} x_n = A_s^n x_0 + \sum_{i=1}^n A_s^{n-i} h_s(p_{i-1}) \\ y_n = \sum_{i=n+1}^\infty A_{cu}^{n+1-i} h_{cu}(p_i) \end{array} \right. \tag{13}$$

for all $n \ge 0$, then it is straightforward to check

$$x_{n+1} = A_s x_n + h_s(x_n, y_n) \text{ and } y_n = A_{cu}^{-1} y_{n+1} + h_{cu}(x_{n+1}, y_{n+1})$$

hold for all $n \ge 0$, and because of (1) the sequence is an orbit of f. Therefore, $p \in W^{\mathrm{s}}$ iff $\gamma \in S_\alpha$ with $p_0 = p$ is a fixed point of T for which the identity (9) holds. □

Lemma 2. *There is a Lipschitz continuous function $\phi_{cu} \in C^{0,1}(\mathbb{E}^s, \mathbb{E}^{cu})$ so that*

$$W^{\mathrm{s}} = \mathrm{graph}(\phi_{cu}). \tag{14}$$

Also W^{s} and ϕ_{cu} are independent of any two different choices in α.

Proof. By Lemma 1, we know that $p \in W^{\mathrm{s}}$ if and only if p is the initial point of a sequence $\gamma \in S_\alpha$ which is a fixed point of the map T defined by (6) and (9) holds. To show the existence of such a fixed point, we will consider T as a parameterized map by $x_0 \in \mathbb{E}^s$ and show that $T(\cdot, x_0) : S_\alpha \to S_\alpha,\ x_0 \in \mathbb{E}^s$, is a uniform contraction. Specifically, let γ, γ' and $\bar{\gamma} = T(\gamma, x_0), \bar{\gamma}' = T(\gamma', x_0)$, then we have

$$\begin{aligned} \|\bar{x}_n - \bar{x}_n'\| &\leq \textstyle\sum_{i=1}^n \|A_s^{n-i}[h_s(p_{i-1}) - h_s(p_{i-1}')]\| \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} L\|p_{i-1} - p_{i-1}'\| \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} L\alpha^{i-1}\|\gamma - \gamma'\|_\alpha \\ &\leq \tfrac{L}{\alpha-\nu}\alpha^n\|\gamma - \gamma'\|_\alpha \end{aligned} \tag{15}$$

and

$$\begin{aligned} \|\bar{y}_n - \bar{y}_n'\| &\leq \textstyle\sum_{i=n+1}^\infty \|A_{cu}^{n+1-i}[h_{cu}(p_i) - h_{cu}(p_i')]\| \\ &\leq \textstyle\sum_{i=n+1}^\infty \beta^{i-n-1} L\|p_i - p_i'\| \\ &\leq \textstyle\sum_{i=n+1}^\infty \beta^{i-n-1} L\alpha^i\|\gamma - \gamma'\|_\alpha \\ &\leq \tfrac{L\alpha}{1-\alpha\beta}\alpha^n\|\gamma - \gamma'\|_\alpha. \end{aligned} \tag{16}$$

Hence,

$$\|T(\gamma, x_0) - T(\gamma', x_0)\|_\alpha \leq (\tfrac{L}{\alpha-\nu} + \tfrac{L\alpha}{1-\alpha\beta})\|\gamma - \gamma'\|_\alpha$$

showing $T(\cdot, x_0)$ is a uniform contraction in S_α provided

$$\theta := \theta(\alpha) = \tfrac{L}{\alpha-\nu} + \tfrac{L\alpha}{1-\alpha\beta} < 1 \tag{17}$$

which is true for sufficiently small $\|f - Df(\bar{q})\|_1$. Let

$$\gamma^*(x_0) = \{p_n(x_0)\}_{n=0}^\infty, \;\; p_n(x_0) = (x_n(x_0), y_n(x_0)), \; n \geq 0 \tag{18}$$

be the unique fixed point of $T(\cdot, x_0)$ for each $x_0 \in \mathbb{E}^s$. Because $\|A_s^n\| \leq \nu^n < \alpha^n$, $T(\gamma, x_0)$ is Lipschitz continuous in x_0 with

$$\|T(\gamma, x_0) - T(\gamma, {x_0}')\|_\alpha \leq \|x_0 - {x_0}'\|.$$

Thus, by the Uniform Contraction Principle I, $\gamma^*(x_0)$ is Lipschitz continuous with

$$\|\gamma^*(x_0) - \gamma^*({x_0}')\|_\alpha \leq \tfrac{1}{1-\theta}\|x_0 - {x_0}'\|. \tag{19}$$

Define

$$\phi_{cu}(x_0) = y_0(x_0) = \textstyle\sum_{i=1}^\infty A_{cu}^{1-i} h_{cu}(p_i(x_0)), \tag{20}$$

the y-coordinate of the initial point of the fixed point $\gamma^*(x_0)$. Then by (19),

$$\|\phi_{cu}(x_0) - \phi_{cu}({x_0}')\| \leq \|\gamma^*(x_0) - \gamma^*({x_0}')\|_\alpha \leq \tfrac{1}{1-\theta}\|x_0 - {x_0}'\|,$$

proving $\phi_{cu} \in C^{0,1}(\mathbb{E}^s, \mathbb{E}^{cu})$. Since $p = (x_0, y_0) = (x_0, \phi_{cu}(x_0)) \in W^{\mathrm{s}}$, the identity (14) holds.

To show W^{s} and ϕ_{cu} are independent of the choice of α, let α' and α be two different constants satisfying the definition W^{s}. We can re-adjust the adapted norm so that

$$\|A_s\| < \alpha' < \alpha < 1.$$

Then results above show that for sufficiently small $\|f - Df(\bar{q})\|_1$, both $\theta(\alpha')$ and $\theta(\alpha)$ are smaller 1. As a result,

$$W_{\alpha'}^{\mathrm{s}} = \mathrm{graph}(\phi_{cu,\alpha'}), \quad W_\alpha^{\mathrm{s}} = \mathrm{graph}(\phi_{cu,\alpha}).$$

On one hand, $W^{\mathrm{s}}_{\alpha'} \subseteq W^{\mathrm{s}}_{\alpha}$ is automatic because $S_{\alpha'} \subset S_\alpha$ for $\alpha' < \alpha$. On the other hand, because of the uniqueness of the contraction mapping $T(\cdot, x_0)$ on S_α with $S_{\alpha'}$ being a closed subspace of S_α, we must have $\gamma^*_{\alpha'}(x_0) \equiv \gamma^*_\alpha(x_0)$ and $\phi_{cu,\alpha'} \equiv \phi_{cu,\alpha}$. Hence, $W^{\mathrm{s}}_\alpha \subseteq W^{\mathrm{s}}_{\alpha'}$ and

$$W^{\mathrm{s}}_\alpha = W^{\mathrm{s}}_{\alpha'}$$

follows, showing the independence of the definition on any two choices in α. □

Lemma 3. *f is a uniform contraction on W^{s}.*

Proof. Let $p_0 = (x_0, \phi_{cu}(x_0)), p_0' = (x_0', \phi_{cu}(x_0'))$ be two points from W^{s}, and consider their images under f, $p_1 = f(p_0), p_1' = f(p_0')$. Because they are fixed points of T, by (13) we have

$$\begin{aligned}\|x_1 - x_1'\| &\leq \|A_s\|\|x_0 - x_0'\| + \|h_s(p_0) - h_s(p_0')\| \\ &\leq \nu\|x_0 - x_0'\| + L\|p_0 - p_0'\| \\ &\leq (\nu + L)\|p_0 - p_0'\|\end{aligned}$$

and by (19)

$$\begin{aligned}\|y_1 - y_1'\| &\leq \textstyle\sum_{i=2}^{\infty} \|A_{cu}^{2-i}[h_{cu}(p_i(x_0)) - h_{cu}(p_i(x_0')]\| \\ &\leq \textstyle\sum_{i=2}^{\infty} \beta^{i-2} L\|p_i(x_0) - p_i(x_0')\| \\ &\leq L\textstyle\sum_{i=2}^{\infty} \beta^{i-2}\alpha^i\|\gamma^*(x_0) - \gamma^*(x_0')\|_\alpha \\ &\leq \tfrac{L\alpha^2}{1-\alpha\beta}\tfrac{1}{1-\theta}\|x_0 - x_0'\| \\ &\leq \tfrac{L\alpha^2}{1-\alpha\beta}\tfrac{1}{1-\theta}\|p_0 - p_0'\|\end{aligned}$$

implying

$$\|f(p_0) - f(p_0')\| \leq (\nu + L + \tfrac{L\alpha^2}{1-\alpha\beta}\tfrac{1}{1-\theta})\|p_0 - p_0'\|$$

which is a uniform contraction for small L, i.e., for small $\|f - Df(\bar{q})\|_1$. □

Lemma 4. *If $f \in C^k(\mathbb{R}^d)$, then $\phi_{cu} \in C^k(\mathbb{E}^s, \mathbb{E}^{cu})$, and $\mathbb{T}_{\bar{q}}W^{\mathrm{s}} = \mathbb{E}^s$.*

Proof. To show $\phi_{cu}(\cdot)$ is as smooth as f is, it suffices to show the fixed point $\gamma^*(\cdot)$ is as smooth as f. By the Uniform Contraction Principle II, we only need to verify two conditions: (1) $\|D_\gamma T(\gamma, x_0)\|$ is uniformly bounded by a constant smaller than 1; (2) $T \in C^k(S_\alpha \times \mathbb{E}^s, S_\alpha)$.

To show (1), let $\gamma = \{p_n\}, v = \{v_n\} \in S_\alpha$, and formally differentiate (6). Then $D_\gamma T(\gamma, x_0)v$ needs to be as below in components:

$$\begin{cases} [D_\gamma T(\gamma, x_0)v]_{n,\, s} = \sum_{i=1}^{n} A_s^{n-i} Dh_s(p_{i-1})v_{i-1} \\ [D_\gamma T(\gamma, x_0)v]_{n,\, cu} = \sum_{i=n+1}^{\infty} A_{cu}^{n+1-i} Dh_{cu}(p_i)v_i \,. \end{cases} \tag{21}$$

By the exactly same estimate as for (15) we have

$$\|[D_\gamma T(\gamma, x_0)v]_{n,\, s}\| \leq \tfrac{L}{\alpha-\nu}\alpha^n\|v\|_\alpha \,.$$

Similarly, by the exactly same estimate as for (16) we have

$$\|[D_\gamma T(\gamma, x_0)v]_{n,\ cu}\| \leq \tfrac{L\alpha}{1-\alpha\beta}\alpha^n \|v\|_\alpha .$$

These estimates imply two conclusions. One, because of the uniform convergence of the second equation, it shows the derivative $D_\gamma T(\gamma, x_0)$ exists. Two, it shows the derivative is a bounded linear map in $L(S_\alpha, S_\alpha)$ whose α-norm

$$\|D_\gamma T(\gamma, x_0)\|_\alpha \leq \theta(\alpha) < 1,$$

is bounded by the same uniform contraction constant $\theta(\alpha)$ from (17).

To show (2), we note first that

$$[D_{x_0} T(\gamma, x_0)]_{n,\ s} = A_s^n, \text{ and } [D_{x_0} T(\gamma, x_0)]_{n,\ cu} = 0.$$

This implies any mixed derivative in γ and x_0 are the zero operators, hence well-defined and exists. So, we only need to show T is C^k separately in γ and x_0. For the latter, the identity above shows

$$\|[D_{x_0} T(\gamma, x_0)]_n\| \leq \|A_s^n\| \leq \alpha^n$$

and $\|D_{x_0} T(\gamma, x_0]\|_\alpha \leq 1$ follows. Also, $D_{x_0}^j T(\gamma, x_0) = 0$, for $2 \leq j \leq k$. Hence, T is C^k in x_0.

Now we show T is C^k in γ, i.e., $D_\gamma^j T(\gamma, x_0)$ exists and is bounded for any $1 \leq j \leq k$. The case of $j = 1$ was done above. For any $2 \leq j \leq k$, $[D_\gamma^j T(\gamma, x_0)]$ is a j-linear form in S_α. To this end, let $v = v^1 \otimes v^2 \otimes \cdots \otimes v^j$ with each $v^\ell \in S_\alpha$. Formally differentiate (6) to get

$$\begin{cases} [D_\gamma^j T(\gamma, x_0)v]_{n,\ s} = \sum_{i=1}^n A_s^{n-i} D^j h_s(p_{i-1}) v_{i-1} \\ [D_\gamma^j T(\gamma, x_0)v]_{n,\ cu} = \sum_{i=n+1}^\infty A_{cu}^{n+1-i} D^j h_{cu}(p_i) v_i \ , \end{cases} \tag{22}$$

where

$$v_i = v_i^1 \otimes v_i^2 \otimes \cdots \otimes v_i^j, \quad v_i^\ell \in \mathbb{R}^d.$$

Similar to the estimate of (15), we have

$$\begin{aligned} \|[D_\gamma^j T(\gamma, x_0)v]_{n,\ s}\| &\leq \sum_{i=1}^n \|A_s^{n-i}\| \|D^j h_s\| \|v_{i-1}\| \\ &\leq \sum_{i=1}^n \nu^{n-i} \|h_s\|_j \Pi_{\ell=1}^j \|v_{i-1}^\ell\| \\ &\leq \|h_s\|_k \sum_{i=1}^n \nu^{n-i} \alpha^{j(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\leq \|h_s\|_k \sum_{i=1}^n \nu^{n-i} \alpha^{(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\leq \tfrac{\|h_s\|_k}{\alpha-\nu} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\alpha. \end{aligned} \tag{23}$$

Similarly, by an exactly same estimate as (16) we can have

$$\begin{aligned} \|[D_\gamma^j T(\gamma, x_0)v]_{n,\ cu}\| &\leq \sum_{i=n+1}^\infty \|A_{cu}^{n+1-i}\| \|D^j h_{cu}\| \|v_i\| \\ &\leq \sum_{i=n+1}^\infty \beta^{i-n-1} \|h_{cu}\|_j \alpha^{ji} \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\leq \|h_{cu}\|_k \beta^{-n-1} \sum_{i=n+1}^\infty (\beta\alpha^j)^i \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\leq \|h_{cu}\|_k \beta^{-n-1} \sum_{i=n+1}^\infty (\beta\alpha)^i \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\leq \tfrac{\|h_{cu}\|_k \alpha}{1-\alpha\beta} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\alpha. \end{aligned} \tag{24}$$

Combine these two estimates to obtain

$$\|[D_\gamma^j T(\gamma, x_0)]\|_\alpha \leq \|(h_s, h_{cu})\|_k \max\{\tfrac{1}{\alpha-\nu}, \tfrac{\alpha}{1-\alpha\beta}\}.$$

The convergence of the infinite series also shows the derivatives are well-defined. This completes the proof that $T \in C^k(S_\alpha \times \mathbb{E}^s, S_\alpha)$.

Finally, for the derivative of ϕ_{cu} as the fixed point for T, we have from (20)

$$D\phi_{cu}(x_0) = \textstyle\sum_{i=1}^\infty A_{cu}^{1-i} Dh_{cu}(p_i(x_0)) Dp_i(x_0).$$

Because $Dh_{cu}(0) = 0$ and $\gamma^*(0) = \{p_n(0) = 0 : n \geq 0\}$ is the trivial fixed point corresponding to the fixed point $\bar{q} \sim 0$, we have

$$\phi_{cu}(0) = 0 \text{ and } D\phi_{cu}(0) = 0,$$

showing that the tangent space of W^{s} at the fixed point is the stable eigenspace $\mathbb{R}^{d_s} \cong \mathbb{E}^s$. This completes the proof. □

Definition 2. *Let $\bar{q}$ be a nonsingular fixed point of a differentiable mapping f in $\mathbb{R}^d$ and β be any constant satisfying*

$$1 < \beta < \min\{|\sigma^u|\}.$$

The unstable manifold of the fixed point $\bar{q}$ for f is

$$W^{\mathrm{s}} = \{p : \{\beta^n [f^{-n}(p) - \bar{q}]\}_{n=0}^\infty \text{ is a bounded sequence.}\}$$

By applying the theorem above to f^{-1} we can prove the following theorem.

Theorem 2 (Unstable Manifold Theorem)**.** *Let $\bar{q}$ be a nonsingular fixed point of a continuously differentiable map f in $\mathbb{R}^d$ with splitting $\mathbb{R}^d \cong \mathbb{E}^{cs} \oplus \mathbb{E}^u$. Then for sufficiently small $\delta > 0$, $\|f - Df(\bar{q})\|_1 < \delta$ implies the definition of W^{u} is independent of any two different choices in the constant β. Also W^{u} is the graph of a C^1 function $\phi_{cs} : \mathbb{E}^u \to \mathbb{E}^{cs}$*

$$W^{\mathrm{u}} = \mathrm{graph}(\phi_{cs}),$$

and the tangent space of W^{u} at the fixed point is the unstable eigenspace

$$\mathbb{T}_{\bar{q}} W^{\mathrm{u}} = \mathbb{E}^u.$$

Moreover, f^{-1} is a contraction on W^{u}. Furthermore, if f is $C^k,\ k \geq 1$, and all its derivatives $D^j f$, $1 \leq j \leq k$, are bounded, then ϕ_{cs} is also C^k with bounded derivatives.

Theorem 3 (Local Stable and Local Unstable Manifold Theorem)**.** *Let $\bar{q}$ be a nonsingular fixed point of a continuously differentiable map f in $\mathbb{R}^d$ and let $\mathbb{E}^s$, $\mathbb{E}^u$ be the stable, respectively, the unstable eigenspace at $\bar{q}$ for the linearization $Df(\bar{q})$. Let α, β be any constants satisfying*

$$\max\{|\sigma^s|\} < \alpha < 1 < \beta < \min\{|\sigma^u|\}.$$

Then there is a small neighborhood $N_r(\bar{q})$ and two differentiable functions ϕ_{cu} : $N_r(\bar{q}) \cap \mathbb{E}^s \to \mathbb{E}^{cu}$, $\phi_{cs} : N_r(\bar{q}) \cap \mathbb{E}^u \to \mathbb{E}^{cs}$, so that the local stable and local unstable manifolds

$$W^{\mathrm{s}}_{\mathrm{loc}}(\bar{q}) := \mathrm{graph}(\phi_{cu}), \quad W^{\mathrm{u}}_{\mathrm{loc}}(\bar{q}) := \mathrm{graph}(\phi_{cs})$$

satisfy the following properties

(i) $W^{\mathrm{s}}_{\mathrm{loc}} = \{p \in N_r : \lim_{n\to\infty} f^n(p) = \bar{q}$ *at rate* $\alpha^n\}$

(ii) $W^{\mathrm{u}}_{\mathrm{loc}} = \{p \in N_r : \lim_{n\to\infty} f^{-n}(p) = \bar{q}$ *at rate* $\beta^{-n}\}$

(iii) f *is a contraction on* $W^{\mathrm{s}}_{\mathrm{loc}}$*, and* f^{-1} *is a contraction on* $W^{\mathrm{u}}_{\mathrm{loc}}$ *.*

(iv) $\mathbb{T}_{\bar{q}} W^{\mathrm{s}}_{\mathrm{loc}} = \mathbb{E}^s$, $\mathbb{T}_{\bar{q}} W^{\mathrm{u}}_{\mathrm{loc}} = \mathbb{E}^u$

Moreover, if f is C^k, $k \geq 1$, then both $W^{\mathrm{s}}_{\mathrm{loc}}$ and $W^{\mathrm{u}}_{\mathrm{loc}}$ are C^k manifolds.

Proof. Modify the map f by a C^∞ cut-off function $\rho_r(p - \bar{q})$ to $f \to f(p) = Df(\bar{q})p + \rho_r(p - \bar{q})(f(p) - Df(\bar{q})p)$. Then for sufficiently small r, Theorems 1 and 2 can be applied to the modified map to obtain the maps ϕ_{cu}, ϕ_{cs}. Restrict both to the neighborhood $N_r(\bar{q})$, then the results follow from the theorems. □

Definition 3. *Let $\bar{q}$ be a nonsingular fixed point of a continuously differentiable map f in $\mathbb{R}^d$. The global stable manifold of the fixed point is defined as*

$$W^{\mathrm{s}}_{\mathrm{glb}}(\bar{q}) = \cup_n^\infty f^{-n}(W^{\mathrm{s}}_{\mathrm{loc}}(\bar{q}))$$

and the global unstable manifold is defined as

$$W^{\mathrm{u}}_{\mathrm{glb}}(\bar{q}) = \cup_n^\infty f^{n}(W^{\mathrm{s}}_{\mathrm{loc}}(\bar{q})).$$

A point $\bar{p}$ is called a homoclinic point of a hyperbolic fixed point $\bar{q}$ of f if $\bar{p}$ is an intersection of $W^{\mathrm{s}}_{\mathrm{glb}}(\bar{q})$ and $W^{\mathrm{u}}_{\mathrm{glb}}(\bar{q})$. We note that if the global stable and unstable manifolds intersect transversely, then a horseshoe dynamics arises, and hence f is expected to be chaotic in a neighborhood of the homoclinic orbit.