Stable and Unstable Manifolds I

This notes is about stable and unstable manifolds for hyperbolic fixed points of diffeomorphisms.

Let $\bar{q}$ be a hyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$. Denote

$$\sigma^s = \sigma(Df(\bar{q})) \cap \{|z| < 1\} \text{ and } \sigma^u = \sigma(Df(\bar{q})) \cap \{|z| > 1\}$$

the set of stable eigenvalues and, respectively, the set of unstable eigenvalues of the linearizatoin $Df(\bar{q})$.

Definition 1. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism and $\bar{q}$ be a nonsingular fixed point. The stable manifold of the fixed point $\bar{q}$ for f is*

$$W^{\mathrm{s}} = \{p : \{f^n(p)\}_{n=0}^{\infty} \textit{ is a bounded sequence.}\}$$

The unstable manifold of the fixed point is

$$W^{\mathrm{u}} = \{p : \{f^{-n}(p)\}_{n=0}^{\infty} \textit{ is a bounded sequence.}\}$$

Theorem 1 (Stable Manifold Theorem)**.** *Let $\bar{q}$ be a hyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$ with hyperbolic splitting $\mathbb{R}^d \cong \mathbb{E}^s \oplus \mathbb{E}^u$ for the linearization $Df(\bar{q})$. Then a sufficiently small $\|f - Df(\bar{q})\|_1$ implies W^{s} is the graph of a C^1 function $\phi_u : \mathbb{E}^s \to \mathbb{E}^u$*

$$W^{\mathrm{s}} = \mathrm{graph}(\phi_u),$$

and the tangent space of W^{s} at the fixed point is the stable eigenspace

$$\mathbb{T}_{\bar{q}} W^{\mathrm{s}} = \mathbb{E}^s.$$

Moreover, f is a uniform contraction on W^{s}. In addition, let α be any constant satisfying

$$\max\{|\sigma^s|\} < \alpha < 1,$$

then for any $p \in W^{\mathrm{s}}$ there is a constant R so that

$$\|f^n(p) - \bar{q}\| \le R\alpha^n, \quad \textit{for all } n \ge 0.$$

Furthermore, if f is C^k, $k \ge 1$, and all its derivatives $D^j f$, $1 \le j \le k$, are bounded, then ϕ_u is also C^k with bounded derivatives.

The proof is an application of the Uniform Contraction Principle. The main idea is to construct the center-stable manifold function ϕ_u as part of a fixed point of a uniform contraction map. We will break it up into a few lemmas.

Before doing so, we recall a few facts for the system. We first choose a coordinate system (x, y) for the hyperbolic splitting in which $Df(\bar{q}) \cong \mathrm{diag}(A_s, A_u)$. An adapted norm is also chosen so that for any constant α from the statement of the theorem we can fix two more constants ν and $\bar{\alpha}$ so that the following inequalities hold

$$\|A_s\| < \nu < \alpha < 1 \text{ and } \|{A_u}^{-1}\| < \bar{\alpha} < 1. \tag{1}$$

Second, by the Variation of Parameters Formula Theorem, for sufficiently small $\|f - Df(\bar{q})\|_1$, the map $(\bar{x}, \bar{y}) = f(x, y)$ is equivalent to

$$\begin{cases} \bar{x} = A_s x + h_s(x, y) \\ y = {A_u}^{-1}\bar{y} + h_u(\bar{x}, \bar{y}), \end{cases} \tag{2}$$

and for any orbit, $p_n = (x_n, y_n) = f(x_{n-1}, y_{n-1}),\ n \geq 0$,

$$\begin{cases} x_n = {A_s}^{n-\ell} x_\ell + \sum_{i=\ell+1}^{n} {A_s}^{n-i} h_s(x_{i-1}, y_{i-1}) \\ y_n = {A_u}^{n-m} y_m + \sum_{i=n+1}^{m} {A_u}^{n+1-i} h_u(x_i, y_i). \end{cases} \tag{3}$$

We only need orbits from the stable manifold, $(x_0, y_0) \in W^{\mathrm{s}}$, and fix $\ell = 0$ from now on. Also, by the variation of parameter formula theorem, the functions h_s, h_u are all C^1 satisfying

$$h_s(0,0) = 0,\ Dh_s(0,0) = 0,\ h_u(0,0) = 0,\ Dh_u(0,0) = 0 \tag{4}$$

and they are globally Lipschitz with Lipschitz constants satisfying

$$L = \max\{\mathrm{Lip}(h_s, h_u)\} \to 0 \quad \text{as} \quad \|f - Df(\bar{q})\|_1 \to 0. \tag{5}$$

We will repeatedly use this formula for geometric sequences

$$a + ar + ar^2 + \cdots + ar^{n-1} = \tfrac{a(1-r^n)}{1-r}, \text{ for } r \neq 1$$

and its differentiation formulas in r.

Lemma 1. *Let*

$$\ell^\infty = \{\gamma = \{p_n\}_{n=0}^\infty : p_n = (x_n, y_n) \in \mathbb{R}^d, \sup\{\|p_n\| : n \geq 0\} < +\infty\}$$

be the Banach space of bounded infinite sequences with the supreme norm

$$\|\gamma\|_\infty = \sup\{\|p_n\| : n \geq 0\}.$$

For any $\gamma \in \ell^\infty$, $\gamma = \{p_n\}_{n=0}^\infty$, let $\bar{\gamma} = T(\gamma)$ be defined by the equations below

$$\begin{cases} \bar{x}_n = {A_s}^n x_0 + \sum_{i=1}^{n} {A_s}^{n-i} h_s(p_{i-1}) \\ \bar{y}_n = \sum_{i=n+1}^{\infty} {A_u}^{n+1-i} h_u(p_i). \end{cases} \tag{6}$$

Then $T : \ell^\infty \to \ell^\infty$. More importantly, $p \in W^{\mathrm{s}}$ if and only if the orbit $\gamma_p = \{f^n(p)\}_{n=0}^\infty$ is a fixed point of T with

$$p = (x_0, y_0) = \left(x_0, \textstyle\sum_{i=1}^{\infty} {A_u}^{1-i} h_u(p_i)\right). \tag{7}$$

Proof. Because $h_i(0) = 0$, $\|h_i(p)\| \leq L\|p\|$. Since $\|A^j\| < \nu^j$, $\|{A_u}^{-j}\| < \bar{\alpha}^j$ for any $j \geq 0$, we have

$$\|\bar{x}_n\| \leq \nu^n \|x_0\| + \sum_{i=1}^{n} \nu^{n-i} L \|p_{i-1}\| \leq (\nu + \frac{L}{1-\nu})\|\gamma\|,$$

and

$$\|\bar{y}_n\| \leq \sum_{i=n+1}^{\infty} \bar{\alpha}^{i-n-1} L \|p_i\| \leq \frac{L}{1-\bar{\alpha}}\|\gamma\|,$$

implying

$$\|\bar{\gamma}\| \leq (\nu + \frac{L}{1-\nu} + \frac{L}{1-\bar{\alpha}})\|\gamma\|,$$

and $T : \ell^\infty \to \ell^\infty$ follows.

Now for every $p = (x_0, y_0) \in W^{\mathrm{s}}$, because of $\gamma_p = \{p_n = f^n(p)\}_{n=1}^{\infty} \in \ell^\infty$, the first term of the y_n-equation in (3) tends to 0 as $m \to \infty$. The partial sum term of the y_n-equation converges because of the convergence of the series by the estimate for $\bar{y}_n$ above. Hence, by taking $m \to \infty$ in (3) we obtain

$$\begin{cases} x_n = {A_s}^n x_0 + \sum_{i=1}^{n} {A_s}^{n-i} h_s(p_{i-1}) \\ y_n = \sum_{i=n+1}^{\infty} {A_u}^{n+1-i} h_u(p_i), \end{cases} \tag{8}$$

showing γ_p is a fixed point of T. Conversely, let $\gamma = \{p_n\}_{n=1}^{\infty} \in \ell^\infty$ be a fixed point T, satisfying (8). Then it is straightforward to check

$$x_{n+1} = A_s x_n + h_s(x_n, y_n) \;\text{ and }\; y_n = {A_u}^{-1} y_{n+1} + h_u(x_{n+1}, y_{n+1})$$

hold for all $n \geq 0$, and by (2) the sequence must be an orbit of f, namely, $p_n = f(p_{n-1})$ for all $n \geq 1$. As a result, the initial point $p = (x_0, y_0)$ must be given by (7). □

Lemma 2. *There is a Lipschitz continuous function $\phi_u \in C^{0,1}(\mathbb{E}^s, \mathbb{E}^u)$ so that $\phi_u(0) = 0$ and*

$$W^{\mathrm{s}} = \mathrm{graph}(\phi_u). \tag{9}$$

Proof. By Lemma 1, we know that $p \in W^{\mathrm{s}}$ if and only if p is the initial point of a sequence $\gamma \in \ell^\infty$ which is a fixed point of the map T defined by (6) and (7) holds. To show the existence of such a fixed point, we will consider T as a parameterized map by $x_0 \in \mathbb{E}^s$ and show that $T(\cdot, x_0) : \ell^\infty \to \ell^\infty,\ x_0 \in \mathbb{E}^s$, is a uniform contraction. Specifically, let γ, γ' and $\bar{\gamma} = T(\gamma, x_0), \bar{\gamma}' = T(\gamma', x_0)$. We have

$$\begin{aligned} \|\bar{x}_n - \bar{x}'_n\| &\leq \sum_{i=1}^{n} \|{A_s}^{n-i}[h_s(p_{i-1}) - h_s(p'_{i-1})]\| \\ &\leq \sum_{i=1}^{n} \nu^{n-i} L \|p_{i-1} - p'_{i-1}\| \\ &\leq \tfrac{L}{1-\nu}\|\gamma - \gamma'\|_\infty \end{aligned} \tag{10}$$

and

$$\begin{aligned} \|\bar{y}_n - \bar{y}'_n\| &\leq \sum_{i=n+1}^{\infty} \|{A_u}^{n+1-i}[h_u(p_i) - h_u(p'_i)]\| \\ &\leq \sum_{i=n+1}^{\infty} \bar{\alpha}^{i-n-1} L \|p_i - p'_i\| \\ &\leq \tfrac{L}{1-\bar{\alpha}}\|\gamma - \gamma'\|_\infty. \end{aligned} \tag{11}$$

Hence

$$\|T(\gamma, x_0) - T(\gamma', x_0)\|_\infty \leq L(\tfrac{1}{1-\nu} + \tfrac{1}{1-\bar{\alpha}})\|\gamma - \gamma'\|_\infty.$$

Therefore, for sufficiently small $\|f - Df(\bar{q})\|_1$ we can assume by (5)

$$\theta := L(\tfrac{1}{1-\nu} + \tfrac{1}{1-\bar{\alpha}}) < 1 \tag{12}$$

and $T(\cdot, x_0)$ has a unique fixed point

$$\gamma^*(x_0) = \{p_n(x_0)\}_{n=0}^{\infty}, \;\; p_n(x_0) = (x_n(x_0), y_n(x_0)), \; n \geq 0 \tag{13}$$

for each $x_0 \in \mathbb{E}^s$. Furthermore, since $\|{A_s}^n\| < \nu^n < 1, n \geq 1$, it is straightforward to show $T(\gamma, x_0)$ is Lipschitz continuous in x_0 with

$$\|T(\gamma, x_0) - T(\gamma, {x_0}')\|_\infty \leq \|x_0 - {x_0}'\|.$$

Thus by the Uniform Contraction Principle I, $\gamma^*(x_0)$ is Lipschitz continuous with

$$\|\gamma^*(x_0) - \gamma^*({x_0}')\|_\infty \leq \tfrac{1}{1-\theta}\|x_0 - {x_0}'\|. \tag{14}$$

Define

$$\phi_u(x_0) = y_0(x_0) = \textstyle\sum_{i=1}^{\infty} {A_u}^{1-i} h_u(p_i(x_0)), \tag{15}$$

the y-coordinate of the initial point of the fixed point $\gamma^*(x_0)$. Then by (14),

$$\|\phi_u(x_0) - \phi_u({x_0}')\| \leq \tfrac{1}{1-\theta}\|x_0 - {x_0}'\|,$$

proving $\phi_u : \mathbb{E}^s \to \mathbb{E}^u$ is Lipschitz continuous. Because every orbit from $(x_0, y_0) = (x_0, y_0) \in W^{\mathrm{s}}$ is the fixed point of $T(\cdot, x_0)$ for which $(x_0, y_0) = (x_0, \phi_u(x_0))$, the graph identity (9) holds. Also, since the zero sequence $\gamma_0 = \{0\}$ corresponds to the fixed point $\bar{q}$ which is obviously in W^{s}, we have from (15) and the property $h(0) = 0$ that $\phi_u(0) = 0$. □

Lemma 3. *If $f \in C^k(\mathbb{R}^d)$, then $\phi_u \in C^k(\mathbb{E}^s, \mathbb{E}^u)$, and $\mathbb{T}_{\bar{q}} W^{\mathrm{s}} = \mathbb{E}^s$.*

Proof. To show $\phi_u(\cdot)$ is as smooth as f, it suffices to show the fixed point $\gamma^*(\cdot)$ is as smooth as f. By the Uniform Contraction Principle II, we only need to show $T \in C^k(\ell^\infty \times \mathbb{E}^s, \ell^\infty)$ and $\|D_\gamma T(\gamma, x_0)\|$ is uniformly bounded by a constant smaller than 1.

To show T is C^k in x_0, we note first that

$$[D_{x_0} T(\gamma, x_0)]_{n,\,s} = {A_s}^n, \;\text{ and }\; [D_{x_0} T(\gamma, x_0)]_{n,\,u} = 0.$$

This implies any mixed derivative in γ and x_0 are the zero operators, hence well-defined and exists. So, we only need to show T is C^k separately in γ and x_0. For the latter, the identity above shows

$$\|[D_{x_0} T(\gamma, x_0)]_n\| \leq \|{A_s}^n\| \leq \nu^n < 1$$

and $\|D_{x_0} T(\gamma, x_0]\|_\infty \leq 1$ follows. Also, $D^j_{x_0} T(\gamma, x_0) = 0$, for $2 \leq j \leq k$. Hence, T is C^k in x_0.

T is C^k in γ because all derivatives of f to order k are uniformly bounded. This can be seen from the first derivative of T. In fact, for $\gamma = \{p_n\}, v = \{v_n\} \in \ell^\infty$, $D_\gamma T(\gamma, x_0)v$ is given as below in components:

$$\begin{cases} [D_\gamma T(\gamma, x_0)v]_{n,s} = \sum_{i=1}^{n} {A_s}^{n-i} Dh_s(p_{i-1})v_{i-1} \\ [D_\gamma T(\gamma, x_0)v]_{n,u} = \sum_{i=n+1}^{\infty} {A_u}^{n+1-i} Dh_u(p_i)v_i. \end{cases} \tag{16}$$

The derivative exists because the infinite series converges uniformly for bounded $D(h_s, h_u)$ because $f \in C^k(\mathbb{R}^d)$. Similarly, D^jT exists for any $1 \leq j \leq k$ because $D^j(h_s, h_u)$ are bounded for all $j \leq k$ since $(h_s, h_u) \in C^k(\mathbb{R}^d)$. Furthermore, from the equations above we have the following estimates

$$\|[D_\gamma T(\gamma, x_0)v]_{n,s}\| \leq \tfrac{L}{1-\nu}\|v\|_\infty \text{ and } \|[D_\gamma T(\gamma, x_0)v]_{n,u}\| \leq \tfrac{L}{1-\bar{\alpha}}\|v\|_\infty$$

by the same arguments for the uniform contraction of T in the proof of Lemme 2. Hence,

$$\|D_\gamma T(\gamma, x_0)\|_\infty \leq L(\tfrac{1}{1-\nu} + \tfrac{1}{1-\bar{\alpha}}) < 1$$

for the same contraction constant of T as in (12).

Finally, for the derivative of ϕ_u as the fixed point for T, we have from the second equation of (16) with $n = 0$

$$D\phi_u(x_0) = \sum_{i=1}^{\infty} {A_u}^{1-i} Dh_u(p_i(x_0))Dp_i(x_0).$$

Because in addition $Dh_u(0) = 0$, $p_i(0) = (0, 0)$ for all $i \geq 0$, we have

$$D\phi_u(0) = 0,$$

showing that the tangent space of W^{s} at the fixed point is the stable eigenspace $\mathbb{R}^{d_s} \cong \mathbb{E}^s$. This completes the proof. □

Lemma 4. *f is a uniform contraction on W^{s}.*

Proof. Let $p_0 = (x_0, \phi_u(x_0)), p_0' = (x_0', \phi_u(x_0'))$ be two points from W^{s}, and consider their images under f, $p_1 = f(p_0), p_1' = f(p_0')$. Because they are fixed points of T, by (8) we have

$$\begin{aligned} \|x_1 - x_1'\| &\leq \|A_s\|\|x_0 - x_0'\| + \|h_s(p_0) - h_s(p_0')\| \\ &\leq \nu\|x_0 - x_0'\| + L\|p_0 - p_0'\| \\ &\leq (\nu + L)\|p_0 - p_0'\| \end{aligned}$$

and by (14)

$$\begin{aligned} \|y_1 - y_1'\| &\leq \sum_{i=2}^{\infty} \|{A_u}^{2-i}[h_u(p_i(x_0)) - h_u(p_i(x_0'))]\| \\ &\leq \sum_{i=2}^{\infty} \bar{\alpha}^{i-2} L\|p_i - p_i'\| \\ &\leq L\sum_{i=2}^{\infty} \bar{\alpha}^{i-2}\|\gamma^*(x_0) - \gamma^*(x_0')\|_\infty \\ &\leq \tfrac{L}{1-\bar{\alpha}}\tfrac{1}{1-\theta}\|x_0 - x_0'\| \\ &\leq \tfrac{L}{1-\bar{\alpha}}\tfrac{1}{1-\theta}\|p_0 - p_0'\| \end{aligned}$$

implying

$$\|f(p_0) - f(p_0')\| \leq (\nu + L + \tfrac{L}{1-\bar{\alpha}}\tfrac{1}{1-\theta})\|p_0 - p_0'\|$$

which is a uniform contraction for small L, i.e., for small $\|f - Df(\bar{q})\|_1$. □

Lemma 5. *Let α be a fixed constant satisfying,*

$$\max\{|\sigma^s|\} < \alpha < 1.$$

Then for any $p \in W^{\mathrm{s}}$ there is a constant R so that

$$\|f^n(p) - \bar{q}\| \leq R\alpha^n, \quad \textit{for all } n \geq 0.$$

Proof. For the parameter α assume the adapted norm satisfies (1). Let

$$S_\alpha := \{\gamma = \{p_n\}_{n=0}^{\infty} : p_n \in \mathbb{R}^d, \sup\{\alpha^{-n}\|p_n\| : n \geq 0\} < \infty\}. \tag{17}$$

Obviously, S_α is a closed subspace of ℓ^∞. So if we can show $T(\cdot, x_0)$ maps S_α into itself, then the property that $\lim_{n\to\infty} f^n(p) = \bar{q}$ at the required geometric rate of α for any $p \in W^{\mathrm{s}}$ follows.

For any $\gamma \in S_\alpha$, denote it by

$$\|\gamma\|_\alpha = \sup\{\alpha^{-n}\|p_n\| : n \geq 0\}.$$

Then for any $\gamma = \{p_n\}_{n=0}^{\infty} \in S_\alpha$, we have for $\bar{\gamma} = T(\gamma, x_0)$ the following

$$\begin{aligned} \|\bar{x}_n\| &\leq \|{A_s}^n x_0\| + \textstyle\sum_{i=1}^{n} \|{A_s}^{n-i}\| L \|p_{i-1}\| \\ &\leq \nu^n \|x_0\| + L \textstyle\sum_{i=1}^{n} \nu^{n-i}\alpha^{i-1}\|\gamma\|_\alpha \leq \left(\|x_0\| + \frac{L}{\alpha-\nu}\|\gamma\|_\alpha\right)\alpha^n \end{aligned}$$

and similarly,

$$\begin{aligned} \|\bar{y}_n\| &\leq \textstyle\sum_{i=n+1}^{\infty} \|{A_u}^{n+1-i}\| L \|p_i\| \\ &\leq L \textstyle\sum_{i=n+1}^{\infty} \bar{\alpha}^{i-n-1}\alpha^i \|\gamma\|_\alpha \leq \frac{L\alpha}{1-\alpha\bar{\alpha}}\|\gamma\|_\alpha \alpha^n. \end{aligned}$$

Therefore

$$\|(\bar{x}_n, \bar{y}_n)\| \leq R\alpha^n := [\|x_0\| + (\tfrac{1}{\alpha-\nu} + \tfrac{\alpha}{1-\alpha\bar{\alpha}})L\|\gamma\|_\alpha]\alpha^n$$

as required. □

By applying the theorem above to f^{-1} we can prove the following theorem.

Theorem 2 (Unstable Manifold Theorem)**.** *Let $\bar{q}$ be a hyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$. Then a sufficiently small $\|f - Df(\bar{q})\|_1$ implies W^{u} is the graph of a C^1 function $\phi_s : \mathbb{E}^u \to \mathbb{E}^s$*

$$W^{\mathrm{u}} = \mathrm{graph}(\phi_s),$$

and the tangent space of W^{u} at the fixed point is the unstable eigenspace

$$\mathbb{T}_{\bar{q}} W^{\mathrm{u}} = \mathbb{E}^u.$$

Moreover, f^{-1} is a uniform contraction on W^{u}. In addition, let β be any constant satisfying

$$1 < \beta < \min\{|\sigma^u|\},$$

then for any $p \in W^{\mathrm{u}}$ there is a constant R so that

$$\|f^{-n}(p) - \bar{q}\| \leq R\beta^{-n}, \quad \textit{for all } n \geq 0.$$

Furthermore, if f is C^k, $k \geq 1$, and all its derivatives $D^j f$, $1 \leq j \leq k$, are bounded, then ϕ_s is also C^k with bounded derivatives.

Theorem 3 (Local Stable and Local Unstable Manifold Theorem)**.** *Let $\bar{q}$ be a hyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$ and let $\mathbb{E}^s$, $\mathbb{E}^u$ be the stable, respectively, the unstable eigenspace at $\bar{q}$ for the linearization $Df(\bar{q})$. Let α, β be any constants satisfying $\max\{|\sigma^s|\} < \alpha < 1 < \beta < \min\{|\sigma^u|\}$. Then there is a small neighborhood $N_r(\bar{q})$ and two differentiable functions $\phi_u : N_r(\bar{q}) \cap \mathbb{E}^s \to \mathbb{E}^u$, $\phi_s : N_r(\bar{q}) \cap \mathbb{E}^u \to \mathbb{E}^s$, so that the local stable and local unstable manifolds $W^{\mathrm{s}}_{\mathrm{loc}}(\bar{q}) := \mathrm{graph}(\phi_u)$, $W^{\mathrm{u}}_{\mathrm{loc}}(\bar{q}) := \mathrm{graph}(\phi_s)$ satisfy the following properties*

(i) $W^{\mathrm{s}}_{\mathrm{loc}} = \{p \in N_r : \lim_{n\to\infty} f^n(p) = \bar{q}\}$, *and* $\lim_{n\to\infty} f^n(p) = \bar{q}$ *at rate* α^n *for* $p \in W^{\mathrm{s}}_{\mathrm{loc}}$. *f is a uniform contraction on* $W^{\mathrm{s}}_{\mathrm{loc}}$, $f(W^{\mathrm{s}}_{\mathrm{loc}}) \subset W^{\mathrm{s}}_{\mathrm{loc}}$. *And* $\mathbb{T}_{\bar{q}} W^{\mathrm{s}}_{\mathrm{loc}} = \mathbb{E}^s$.

(ii) $W^{\mathrm{u}}_{\mathrm{loc}} = \{p \in N_r : \lim_{n\to\infty} f^{-n}(p) = \bar{q}\}$, *and* $\lim_{n\to\infty} f^{-n}(p) = \bar{q}$ *at rate* β^{-n} *for* $p \in W^{\mathrm{u}}_{\mathrm{loc}}$. *f^{-1} is a uniform contraction on* $W^{\mathrm{u}}_{\mathrm{loc}}$, $f^{-1}(W^{\mathrm{u}}_{\mathrm{loc}}) \subset W^{\mathrm{u}}_{\mathrm{loc}}$. *And* $\mathbb{T}_{\bar{q}} W^{\mathrm{u}}_{\mathrm{loc}} = \mathbb{E}^u$.

Moreover, if f is C^k, $k \geq 1$, then both $W^{\mathrm{s}}_{\mathrm{loc}}$ and $W^{\mathrm{u}}_{\mathrm{loc}}$ are C^k manifolds.

Proof. Modify the map f by a C^∞ cut-off function $\rho_r(p - \bar{q})$ to $f \to f(p) = Df(\bar{q})p + \rho_r(p - \bar{q})(f(p) - Df(\bar{q})(p))$. Then for sufficiently small r, Theorems 1 and 2 can be applied to the modified map to obtain the maps ϕ_u, ϕ_s. Restrict both to the neighborhood $N_r(\bar{q})$, then the results follow from the theorems. □

Definition 2. *Let $\bar{q}$ be a hyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$. The global stable manifold of the fixed point is defined as*

$$W^{\mathrm{s}}_{\mathrm{glb}}(\bar{q}) = \cup_n^\infty f^{-n}(W^{\mathrm{s}}_{\mathrm{loc}}(\bar{q}))$$

and the global unstable manifold is defined as

$$W^{\mathrm{u}}_{\mathrm{glb}}(\bar{q}) = \cup_n^\infty f^{n}(W^{\mathrm{s}}_{\mathrm{loc}}(\bar{q})).$$

A point $\bar{p}$ is called a homoclinic point of a hyperbolic fixed point $\bar{q}$ of f if $\bar{p}$ is an intersection of $W^{\mathrm{s}}_{\mathrm{glb}}(\bar{q})$ and $W^{\mathrm{u}}_{\mathrm{glb}}(\bar{q})$. We note that if the global stable and unstable manifolds intersect transversely, then a horseshoe dynamics arises, and hence f is expected to be chaotic in a neighborhood of the homoclinic orbit.