Stable and Unstable Foliations

Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $J = Df(\bar{q})$, and denote

$$\sigma^s = \sigma(J) \cap \{|z| < 1\}, \sigma^c = \sigma(J) \cap \{|z| = 1\}, \text{ and } \sigma^u = \sigma(J) \cap \{|z| > 1\}$$

the set of stable eigenvalues, center eigenvalues, unstable eigenvalues, respectively, counting multiplicity. Let

$$\sigma^{cs} = \sigma^s \cup \sigma^c, \text{ and } \sigma^{cu} = \sigma^c \cup \sigma^u.$$

Definition 1. *Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$ and $\alpha,\ \beta$ be any constants satisfying*

$$\max\{|\sigma^s|\} < \alpha < 1 < \beta < \min\{|\sigma^u|\}.$$

Let $W^{\rm cs} = \{p : \sup\{\beta^{-n}[f^n(p) - \bar{q}] : n \geq 0\} < \infty\}$ be the center-stable manifold of $\bar{q}$. For every $q \in W^{\rm cs}$ the stable-fiber of q is defined as

$$\mathcal{F}^s(q) = \{p \in W^{\rm cs} : \sup\{\alpha^{-n}[f^n(p) - f^n(q)] : n \geq 0\} < \infty\}$$

and the collection

$$\mathcal{F}^s = \{\mathcal{F}^s(q) : q \in W^{\rm cs}\}$$

is called the stable-foliation of $W^{\rm cs}$.

Notice that the stable-fiber defines an equivalence relation on $W^{\rm cs}$: $q \in \mathcal{F}^s(q)$; $p \in \mathcal{F}^s(q)$ iff $q \in \mathcal{F}^s(p)$ and $\mathcal{F}^s(q) = \mathcal{F}^s(p)$. Also, the foliation is an invariant family with $f(\mathcal{F}^s(q)) = \mathcal{F}^s(f(q))$, and $W^{\rm cs}$ can be filled by fibers through a center manifold as a stem

$$f(\mathcal{F}^s(q)) = \mathcal{F}^s(f(q)), \quad W^{\rm cs} = \cup_{q \in W^{\rm c}} \mathcal{F}^s(q).$$

In addition, the stable manifold is the fiber through $\bar{q}$, $W^{\rm s} = \mathcal{F}^s(\bar{q})$, see Fig.1.

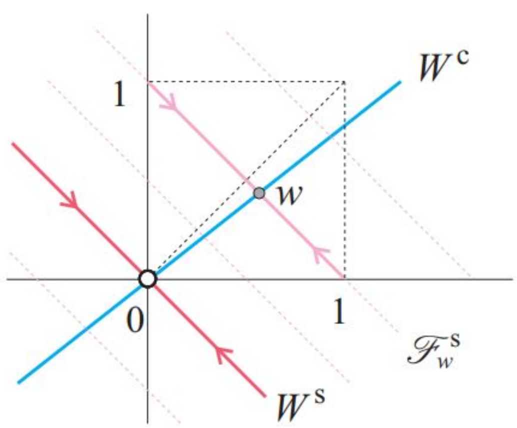

Figure 1. The dynamics of the transition matrix of a Markov process at the trivial fixed point 0 is captured by its foliation through $W^{\rm c}$ which is spanned by the steady-state distribution vector w.

A function f is of $C^{k,1}$ if f is in C^k (itself and all derivatives up to order k are uniformly continuous and bounded in $\mathbb{R}^d$) and its kth derivative is globally Lipschitz continuous. We will use $\|f\|_k$ to denote its C^k norm.

Theorem 1 (Stable Foliation Theorem)**.** *Let $\bar{q}$ be a nonhyperbolic fixed point of a $C^{1,1}$ diffeomorphism f in $\mathbb{R}^d$ with splitting $\mathbb{R}^d \cong \mathbb{E}^s \times \mathbb{E}^c \times \mathbb{E}^u = \mathbb{E}^{cs} \times \mathbb{E}^u$ based at the fixed point. Then a sufficiently small $\|f - Df(\bar{q})\|_1$ implies there is a C^1 function*

$$\psi_{cu} = (\psi_c, \psi_u) : \mathbb{E}^{cs} \times \mathbb{E}^s \to \mathbb{E}^c \times \mathbb{E}^u$$

such that

(i) $q = (q_{cs}, q_u) \in W^{\mathrm{cs}}$ iff $q_u = \psi_u(q_{cs}, q_s)$ with $q_{cs} = (q_s, q_c)$, i.e.,

$$W^{\mathrm{cs}} = \mathrm{graph}(\phi_u) \;\; \textit{with} \; \phi_u(q_{cs}) = \psi_u(q_{cs}, q_s).$$

(ii) $\mathcal{F}^s(q) = \mathrm{graph}(\psi_{cu}(q_{cs}, \cdot))$ for $q \in W^{\mathrm{cs}}$, i.e.,

$$p = (p_s, p_c, p_u) \in \mathcal{F}^s(q) \;\; \textit{if and only if} \; (p_c, p_u) = \psi_{cu}(q_{cs}, p_s).$$

(iii) f is a contraction on each $\mathcal{F}^s(q)$ uniformly for all $q \in W^{\mathrm{cs}}$.

(iv) $\mathcal{F}^s(\bar{q})$ coincides with the stable manifold $\mathcal{F}^s(\bar{q}) = W^{\mathrm{s}}$ and

$$\mathbb{T}_{\bar{q}}\mathcal{F}^s(\bar{q}) \cong \mathbb{E}^s.$$

(v) If f is $C^{k,1}, k \geq 1$, then ψ_{cu} is C^k.

(vi) $\mathcal{F}^s$ is independent of any two different choices in α.

The proof is an application of the Uniform Contraction Principle. The main idea is to construct the stable-foliation function ψ_{cu} as part of a fixed point of a uniform contraction map. We will break it up into a few lemmas. Before doing so, we first recall a few important properties about W^{cs} in the statements below from the proof for the Center Manifold Theorem, assuming the fixed $\bar{q}$ is translated to $0 \in \mathbb{R}^d$.

Proposition 1. *For any $1 < \beta < \min\{|\sigma^u|\}$, let S_β be a Banach space defined by*

$$S_\beta := \{\gamma = \{p_n\}_{n=0}^{\infty} : p_n \in \mathbb{R}^d, \; \sup\{\beta^{-n}\|p_n\| : n \geq 0\} < \infty\}$$

with norm

$$\|\gamma\|_\beta = \sup\{\beta^{-n}\|p_n\| : n \geq 0\}.$$

For any sufficiently small $\|f - Df(\bar{q})\|_1$, the orbit $\gamma_p = \{f^n(p)\}_{n=0}^{\infty}$ of any point $p = (p_{cs}, p_u) \in W^{\mathrm{cs}}$ can be expressed as a function $\gamma_p = \gamma^*(p_{cs})$ for $p_{cs} \in \mathbb{E}^{cs}$ so that $\gamma^* \in C^{k,1}(\mathbb{E}^{cs}, S_\beta)$ if $f \in C^{k,1}(\mathbb{R}^d)$. Moreover, for any $p_{cs}, \; p'_{cs} \in \mathbb{E}^{cs}$

$$\|\gamma^*(p_{cs}) - \gamma^*(p'_{cs})\|_\beta \leq \tfrac{1}{1-\theta_{cs}(\beta)}\|p_{cs} - p'_{cs}\| \tag{1}$$

where $0 < \theta_{cs}(\beta) < 1$ is a uniform contraction constant depending on β. Furthermore, there is a $C^{k,1}$ function $\phi_u : \mathbb{E}^{cs} \to \mathbb{E}^u$ so that the following holds

$$W^{\mathrm{cs}} = \mathrm{graph}(\phi_u), \;\; \phi_u(0) = 0, \;\; \textit{and} \;\; D\phi_u(0) = 0.$$

We also recall that by the Variation of Parameters Formula Theorem (VPF) for splitting $\mathbb{R}^d \cong \mathbb{E}^s \times \mathbb{E}^{cu}$ corresponding to $Df(\bar{q}) \cong \mathrm{diag}(A_s, A_{cu})$, the map $(\bar{x}, \bar{y}) = f(x, y)$ with $(x, y), (\bar{x}, \bar{y}) \in \mathbb{E}^s \times \mathbb{E}^{cu}$ is equivalent to

$$\begin{cases} \bar{x} = A_s x + h_s(x, y) \\ y = {A_{cu}}^{-1} \bar{y} + h_{cu}(\bar{x}, \bar{y}), \end{cases} \tag{2}$$

and for any orbit, $q_n = (x_n, y_n) = f(x_{n-1}, y_{n-1})$, and $n \geq 0$

$$\begin{cases} x_n = {A_s}^n x_0 + \sum_{i=1}^{n} {A_s}^{n-i} h_s(q_{i-1}) \\ y_n = {A_{cu}}^{n-m} y_m + \sum_{i=n+1}^{m} {A_{cu}}^{n+1-i} h_{cu}(q_i). \end{cases} \tag{3}$$

Here, the functions $h_s,\ h_{cu}$ are defined by f and are as smooth as f, satisfying

$$h_s(0) = 0,\ Dh_s(0) = 0,\ h_{cu}(0) = 0,\ Dh_{cu}(0) = 0. \tag{4}$$

They are globally Lipschitz and their Lipschitz constant can be taken to be

$$L = \|D(h_s, h_{cu})\|_0 \to 0 \quad \text{as} \quad \|f - Df(\bar{q})\|_1 \to 0. \tag{5}$$

The result above holds for sufficiently small $\|f - Df(\bar{q})\|_1$.

Associated with h_i, we will need the following functions throughout

$$g_i(q, \delta p) = h_i(q + \delta p) - h_i(q), \ \text{ for } \ i = s, cu. \tag{6}$$

Because $h_i \in C^{k,1}$ so is $g_i \in C^{k,1}$ satisfying

$$g_i(0,0) = 0,\ D_p g_i(0,0) = 0,\ D_{\delta p} g_i(0,0) = 0,\ \text{for } \ i = s, cu. \tag{7}$$

More importantly, all derivatives in q satisfy

$$D_q^j g_i(q, 0) = 0, \ \text{ for } \ 0 \leq j \leq k, \ \text{and } \ i = s, cu. \tag{8}$$

To save notation, we will use the same notation for Lipschitz constants of both h_i and g_i

$$L = \max\{\|D(h_s, h_{cu})\|_0, \|D(g_s, g_{cu})\|_0\} \to 0 \quad \text{as} \quad \|f - Df(\bar{q})\|_1 \to 0. \tag{9}$$

Since $g_i \in C^{k,1}$, we will denote by $L_1, L_2, \ldots, L_k$ the Lipschitz constants for $D_q g_i, D_q^2 g_i, \ldots, D_q^k g_i$, respectively. Together with the fact that $D_q^j g_i(q, 0) = 0$ we have

$$\|D_q^j g_i(q, \delta p)\| \leq L_j \|\delta p\| \ \text{ for } \ 0 \leq j \leq k, \ \text{and } \ i = s, cu. \tag{10}$$

Unlike L which can be made as small as possible by making $\|f - Df(\bar{q})\|_1$ small, these constants L_j are not necessarily small.

We will repeatedly use this formula for geometric sequences

$$a + ar + ar^2 + \cdots + ar^{n-1} = \tfrac{a(1-r^n)}{1-r}, \text{ for } r \neq 1$$

and its differentiation formulas in r. We will denote throughout

$$\gamma_p = \{p_n = f^n(p)\}_{n=0}^{\infty}$$

the orbit of f with the initial point p, for which $p_0 = p$. The proof now consists of a sequence of lemmas below.

Lemma 1. *For any parameter α satisfying* $\max\{|\sigma^s|\} < \alpha < 1$*, let*

$$\Delta S_\alpha := \{\delta\gamma = \{\delta p_n\}_{n=0}^\infty : \delta p_n \in \mathbb{R}^d, \sup\{\alpha^{-n}\|\delta p_n\| : n \geq 0\} < \infty\} \tag{11}$$

with norm

$$\|\delta\gamma\|_\alpha = \sup\{\alpha^{-n}\|\delta p_n\| : n \geq 0\}.$$

For any $q \in W^{\rm cs}$ with $\gamma_q = \{q_n\}$ and $\delta p = \{\delta p_n\} \in \Delta S_\alpha$, let $\overline{\delta\gamma} = T(\delta\gamma)$ be defined by the equations below

$$\begin{cases} \overline{\delta x}_n = {A_s}^n \delta x_0 + \sum_{i=1}^n {A_s}^{n-i} g_s(q_{i-1}, \delta p_{i-1}) \\ \overline{\delta y}_n = \sum_{i=n+1}^\infty {A_{cu}}^{n+1-i} g_{cu}(q_i, \delta p_i) \end{cases} \tag{12}$$

Then $\overline{\delta\gamma} \in \Delta S_\alpha$ with

$$\|\overline{\delta\gamma}\|_\alpha \leq \|\delta x_0\| + \frac{L\|\delta\gamma\|_\alpha}{\alpha-\nu} + \frac{\alpha L\|\delta\gamma\|_\alpha}{1-\alpha\beta}, \tag{13}$$

where ν, β are fixed constants satisfying

$$\max\{|\sigma^s|\} < \nu < \alpha < 1 < \beta < 1/\alpha.$$

More importantly, $p \in \mathcal{F}^s(q)$ if and only if the orbit difference $\delta\gamma = \gamma_p - \gamma_q$ is a fixed point of T, i.e., $p = q + \delta p$ with $\delta p = (\delta x_0, \delta y_0)$ the initial point of $\delta\gamma$, and specifically,

$$p = (p_s, p_c, p_u) = (q_s, q_c, \phi_u(q_s, q_c)) + \left(\delta x_0, \sum_{i=1}^\infty {A_{cu}}^{1-i} g_{cu}(q_i, \delta p_i)\right). \tag{14}$$

Proof. We first show that T is well-defined together with the bound estimate. We begin by making $1 < \beta$ sufficiently close to 1 and fixing an adapted norm so that the following conditions hold

$$\|A_s\| < \nu < \alpha < 1 \text{ and } \|{A_{cu}}^{-1}\| < \beta < \tfrac{1}{\alpha}. \tag{15}$$

We now demonstrate $\overline{\delta\gamma} = \{(\overline{\delta x}_n, \overline{\delta y}_n)\} \in \Delta S_\alpha$. Because $g_i(q, 0) = 0$ and $\|g_i(q, \delta p)\| \leq L\|\delta p\|$ from (8,10) we have for $\overline{\delta x}_n$

$$\begin{aligned} \|\overline{\delta x}_n\| &\leq \|{A_s}^n\|\|\delta x_0\| + \sum_{i=1}^n \|{A_s}^{n-i} g_s(q_{i-1}, \delta p_{i-1})\| \\ &\leq \nu^n\|\delta x_0\| + \sum_{i=1}^n \nu^{n-i} L\alpha^{i-1}\|\delta\gamma\|_\alpha \\ &= \nu^n\|\delta x_0\| + L\|\delta\gamma\|_\alpha \frac{\alpha^n-\nu^n}{\alpha-\nu} \\ &\leq (\|\delta x_0\| + \frac{L\|\delta\gamma\|_\alpha}{\alpha-\nu})\alpha^n. \end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned} \|\overline{\delta y}_n\| &\leq \sum_{i=n+1}^\infty \|{A_{cu}}^{n+1-i} g_{cu}(q_i, \delta p_i)\| \\ &\leq \sum_{i=n+1}^\infty \beta^{i-n-1} L\alpha^i\|\delta\gamma\|_\alpha \\ &= \beta^{-n-1} L\|\delta\gamma\|_\alpha \frac{(\alpha\beta)^{n+1}}{1-\alpha\beta} \\ &= \frac{\alpha L\|\delta\gamma\|_\alpha}{1-\alpha\beta}\alpha^n. \end{aligned} \tag{17}$$

Hence, the estimate (13) holds. This shows that the infinite series converges uniformly and that T is well-defined, mapping ΔS_α into itself.

Next, we show the last part of the lemma. First, for $p \in \mathcal{F}^s(q)$, both orbits γ_p, γ_q are in S_β, and the orbit difference

$$\delta\gamma = \gamma_p - \gamma_q = \{\delta p_n : \delta p_n = p_n - q_n, n \geq 0\} \tag{18}$$

is in ΔS_α by definition. By the VPF (3), $\delta\gamma$ satisfies

$$\begin{cases} \delta x_n = {A_s}^n \delta x_0 + \sum_{i=1}^n {A_s}^{n-i} g_s(q_{i-1}, \delta p_{i-1}) \\ \delta y_n = {A_{cu}}^{n-m} \delta y_m + \sum_{i=n+1}^m {A_{cu}}^{n+1-i} g_{cu}(q_i, \delta p_i) \end{cases}$$

Because $\|\delta y_m\| \leq \alpha^m \|\delta\gamma\|_\alpha$ and $\|{A_{cu}}^{n-m}\| \leq \beta^{m-n}$ and $\alpha\beta < 1$, the first term in the y_n-equation above converges to 0 as $m \to \infty$. The estimate (17) also shows the partial sum of the y_n-equation converges uniformly. Therefore the limit as $m \to \infty$ exists for the y_n-equation and the limit is exactly the y_n-equation for the map T. Hence, $\delta\gamma$ is a fixed point of T.

Conversely, assume $\delta\gamma = \{(\delta x_n, \delta y_n)\}$ is a fixed point of T for a given γ_q from W^{cs}. It is straightforward to verify

$$\begin{cases} \delta x_n = A_s \delta x_{n-1} + g_s(q_{n-1}, \delta p_{n-1}) \\ \delta y_n = {A_{cu}}^{-1} \delta y_{n+1} + g_{cu}(q_{n+1}, \delta p_{n+1}). \end{cases}$$

Denote $p_n = q_n + \delta p_n,\ p_n = (x_n, y_n),\ q_n = (x_{q,n}, y_{q,n})$. Then because γ_q is an orbit it satisfies

$$\begin{cases} x_{q,n} = A_s x_{q,n-1} + h_s(q_{n-1}) \\ y_{q,n} = {A_{cu}}^{-1} y_{q,n+1} + h_{cu}(q_{n+1}). \end{cases}$$

Sum up these two equations component by component to obtain

$$\begin{cases} x_n = A_s x_{n-1} + h_s(p_{n-1}) \\ y_n = {A_{cu}}^{-1} y_{n+1} + h_{cu}(p_{n+1}), \end{cases}$$

which shows $\gamma = \{p_n\} = \gamma_q + \delta\gamma$ must be an orbit of f, $\gamma = \gamma_{p_0}$. Since $\gamma_q \in S_\beta$ and $\delta\gamma \in \Delta S_\alpha \subset S_\beta$, we must have $\gamma_{p_0} \in S_\beta$. Hence, the initial point, p_0, of γ_{p_0} is in W^{cs} and in $\mathcal{F}^s(q)$ by definition. Last, the identity (14) follows by writing out the initial point of γ_{p_0}. □

Lemma 2. *Let $\phi_u \in C^1(\mathbb{E}^{cs}, \mathbb{E}^u)$ be the function whose graph is W^{cs}. Then there is a function $\psi_{cu} = (\psi_c, \psi_u) : \mathbb{E}^{cs} \times \mathbb{E}^s \to \mathbb{E}^c \times \mathbb{E}^u$ so that for all $w = (w_s, w_c) \in \mathbb{E}^{cs}$,*

$$\phi_u(w) = \psi_u(w, w_s)$$

and for every $q \in W^{\mathrm{cs}}$ with $q = (w, \phi_u(w))$

$$\mathcal{F}^s(w) := \mathcal{F}^s(q) = \mathrm{graph}(\psi_{cu}(w, \cdot)). \tag{19}$$

Moreover, the definition of $\mathcal{F}^s$ is independent of any two different choices in α.

Proof. By Lemma 1, $p \in \mathcal{F}^s(q)$ iff $p = q + \delta p_0$ with δp_0 the initial point of a fixed point $\delta\gamma = \{\delta p_n\}_{n\geq 0}$ of the map T from its proof. We already know q is parameterized by $w \in \mathbb{E}^{cs}$ by $q = (w, \phi_u(w))$. We only need to show δp_0 exists and is parameterized by w and its $\mathbb{E}^s$-coordinate δx_0. In fact, if that is true, then the function ψ_{cu} must be defined from the identity (14) as below

$$(p_s, p_c, p_u) = (x_0, \psi_c(w, x_0), \psi_u(w, x_0)) := (w_s, w_c, \phi_u(w)) + (\delta x_0, \delta y_0(w, \delta x_0)) \tag{20}$$

where $x_0 = p_s,\ \delta x_0 = p_s - q_s = x_0 - w_s$, and

$$\psi_{cu}(w, x_0) = (w_c, \phi_u(w)) + \sum_{i=1}^{\infty} {A_{cu}}^{1-i} g_{cu}(q_i(w), \delta p_i(w, x_0 - w_s)) \ .$$

Assuming the fixed point, denoted by $\delta\gamma^*$, is unique for T, then we see the zero sequence $\delta\gamma^* = \{0\}$ is a trivial fixed point if $\delta x_0 = 0$. As a consequence, we get

$$\psi_u(w, w_s) = \phi_u(w) + \delta p_{0,u}(w, 0) = \phi_u(w),$$

the inclusion of W^{cs}. Definition (20) obviously shows (19). Therefore, it is only left to show the existence and uniqueness of fixed point of T for each w and their independence on any two choices in α.

To this end, we will consider T as a parameterized map $T : \Delta S_\alpha \times \mathbb{E}^{cs} \times \mathbb{E}^s \to \Delta S_\alpha$ with $\overline{\delta\gamma} = T(\delta\gamma, w, \delta x_0)$ being defined by (12) as below

$$\begin{cases} \overline{\delta x}_n = {A_s}^n \delta x_0 + \sum_{i=1}^{n} {A_s}^{n-i} g_s(q_{i-1}(w), \delta p_{i-1}) \\ \overline{\delta y}_n = \sum_{i=n+1}^{\infty} {A_{cu}}^{n+1-i} g_{cu}(q_i(w), \delta p_i) \end{cases} \tag{21}$$

We first show T is a uniform contraction. By the proof of Lemma 1, $T(\cdot, w, \delta x_0)$ maps ΔS_α into ΔS_α. For its uniform contraction, let $\delta\gamma, \delta\gamma' \in \Delta S_\alpha$ and $\overline{\delta\gamma} = T(\delta\gamma, w, \delta x_0), \overline{\delta\gamma}' = T(\delta\gamma', w, \delta x_0)$. Then we have

$$\begin{aligned} \|\overline{\delta x}_n - \overline{\delta x}_n{}'\| &\leq \sum_{i=1}^{n} \|{A_s}^{n-i}[g_s(q_{i-1}(w), \delta p_{i-1}) - g_s(q_{i-1}(w), \delta p'_{i-1})]\| \\ &\leq \sum_{i=1}^{n} \nu^{n-i} L \|\delta p_{i-1} - \delta p'_{i-1}\| \\ &\leq \sum_{i=1}^{n} \nu^{n-i} L \alpha^{i-1} \|\delta\gamma - \delta\gamma'\|_\alpha \\ &\leq \tfrac{L}{\alpha-\nu} \alpha^n \|\delta\gamma - \delta\gamma'\|_\alpha \end{aligned} \tag{22}$$

and

$$\begin{aligned} \|\overline{\delta y}_n - \overline{\delta y}_n{}'\| &\leq \sum_{i=n+1}^{\infty} \|{A_{cu}}^{n+1-i}[g_{cu}(q_i(w), \delta p_i) - g_{cu}(q_i(w), \delta p'_i)]\| \\ &\leq \sum_{i=n+1}^{\infty} \beta^{i-n-1} L \|\delta p_i - \delta p'_i\| \\ &\leq \sum_{i=n+1}^{\infty} \beta^{i-n-1} \alpha^i \|\delta\gamma - \delta\gamma'\|_\alpha \\ &\leq \tfrac{L\alpha}{1-\alpha\beta} \alpha^n \|\delta\gamma - \delta\gamma'\|_\alpha. \end{aligned} \tag{23}$$

Hence,

$$\|T(\delta\gamma, w, \delta x_0) - T(\delta\gamma', w, \delta x_0)\|_\alpha \leq (\tfrac{L}{\alpha-\nu} + \tfrac{L\alpha}{1-\alpha\beta}) \|\delta\gamma - \delta\gamma'\|_\alpha$$

showing $T(\cdot, w, \delta x_0)$ is a uniform contraction in ΔS_α provided

$$\theta := \theta(\alpha) = \tfrac{L}{\alpha-\nu} + \tfrac{L\alpha}{1-\alpha\beta} < 1 \tag{24}$$

which is true for sufficiently small $\|f - Df(\bar{q})\|_1$.

Notice that the existence and uniqueness proof of $\delta\gamma^*$ above shows that for any

$$\|A_s\| < \alpha' < \alpha,$$

as long as

$$\theta(\alpha'),\ \theta(\alpha) < 1$$

$T(\cdot, w, \delta x_0)$ has a unique fixed point in $\Delta S_{\alpha'}$ and ΔS_α. But since $\Delta S_{\alpha'}$ is a closed subspace of ΔS_α, the unique fixed point $\delta\gamma^*(w, \delta x_0)$ is in both $\Delta S_{\alpha'}$ and ΔS_α. This shows the independence of $\mathcal{F}^s$ on any two choices in α. □

Lemma 3. *The foliation function ψ_{cu} is Lipschitz continuous.*

Proof. Notice from its definition (20) that we only need to show δp_0 is Lipschitz for which it suffices to show the unique fixed point $\delta\gamma^*$ of T from Lemma 2 is Lipschitz since δp_0 is only a point of the sequence $\delta\gamma^*$. To begin, let

$$\delta\gamma^*(w, \delta x_0) = \{\delta p_n(w, \delta x_0) = (\delta x_n(w, \delta x_0), \delta y_n(w, \delta x_0))\}_{n=0}^{\infty} \tag{25}$$

be the unique fixed point of $T(\cdot, w, \delta x_0)$ for each $(w, \delta x_0) \in \mathbb{E}^{cs} \times \mathbb{E}^s$. We first fix constant α' and make β closer to 1 if necessary so that the following relations hold

$$\|A_s\| < \nu < \alpha' < \alpha'\beta < \alpha < 1 < \beta < \tfrac{1}{\alpha} < \tfrac{1}{\nu}. \tag{26}$$

We will treat the fixed point $\delta\gamma^*(w, \delta x_0)$ in both $\Delta S_{\alpha'}$ and ΔS_α. We will also use the estimate below

$$\|\delta\gamma^*(w, \delta x_0)\|_\alpha \leq \tfrac{1}{1-\theta}\|\delta x_0\| \tag{27}$$

which follows from the estimate (13) of Lemma 1.

We are now ready to show $\delta\gamma^*$ is Lipschitz in w and δx_0 respectively. Since $T(\delta\gamma, w, \delta x_0)$ is Lipschitz continuous in δx_0 with

$$\|T(\delta\gamma, w, \delta x_0) - T(\delta\gamma, w, \delta x_0{}')\|_\alpha \leq \|\delta x_0 - \delta x_0{}'\|,$$

because $\|A_s{}^n\| < \nu^n$, we have by the Uniform Contraction Principle I that $\delta\gamma^*$ is $C^{0,1}$ in δx_0 with

$$\|\delta\gamma^*(w, \delta x_0) - \delta\gamma^*(w, \delta x_0{}')\|_\alpha \leq \tfrac{1}{1-\theta}\|\delta x_0 - \delta x_0{}'\|. \tag{28}$$

To show the fixed point is $C^{0,1}$ in w, notice first from (10) that for $i = s, cu$,

$$\|g_i(q, \delta p) - g_i(q', \delta p)\| \leq \|D_q g_i(\cdot, \delta p)\|_0 \|q - q'\| \leq L_1\|\delta p\|\|q - q'\|. \tag{29}$$

This estimate together with the comments above on α' and α imply

$$\begin{aligned} \|\overline{\delta x}_n - \overline{\delta x}_n{}'\| &\leq \textstyle\sum_{i=1}^n \|A_s{}^{n-i}[g_s(q_{i-1}(w), \delta p_{i-1}) - g_s(q_{i-1}(w'), \delta p_{i-1})]\| \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} L_1 \|\delta p_{i-1}\| \|q_{i-1}(w) - q_{i-1}(w')\| \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} L_1 \alpha'^{i-1} \|\delta\gamma\|_{\alpha'} \beta^{i-1} \|\gamma^*(w) - \gamma^*(w')\|_\beta \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} L_1 \alpha^{i-1} \|\delta\gamma\|_\alpha \|\gamma^*(w) - \gamma^*(w')\|_\beta \\ &\leq \tfrac{L_1}{\alpha-\nu}\alpha^n \|\delta\gamma\|_\alpha \|\gamma^*(w) - \gamma^*(w')\|_\beta \end{aligned} \tag{30}$$

since $\alpha'\beta < \alpha$, $\|\delta\gamma\|_{\alpha'} \le \|\delta\gamma\|_{\alpha}$. And

$$\begin{aligned}
\|\overline{\delta y}_n - \overline{\delta y}_n{}'\| &\le \textstyle\sum_{i=n+1}^{\infty} \|A_{cu}{}^{n+1-i}[g_{cu}(q_i(w),\delta p_i) - g_{cu}(q_i(w'),\delta p_i)]\| \\
&\le \textstyle\sum_{i=n+1}^{\infty} \beta^{i-n-1} L_1 \|\delta p_i\| \|q_i(w) - q_i(w')\| \\
&\le \textstyle\sum_{i=n+1}^{\infty} \beta^{i-n-1} L_1 \alpha'^i \|\delta\gamma\|_{\alpha'} \beta^i \|\gamma^*(w) - \gamma^*(w')\|_{\beta} \\
&\le \textstyle\sum_{i=n+1}^{\infty} \beta^{i-n-1} L_1 \alpha^i \|\delta\gamma\|_{\alpha} \|\gamma^*(w) - \gamma^*(w')\|_{\beta} \\
&\le \tfrac{L_1\alpha}{1-\alpha\beta} \alpha^n \|\delta\gamma\|_{\alpha} \|\gamma^*(w) - \gamma^*(w')\|_{\beta}.
\end{aligned} \tag{31}$$

Therefore,

$$\|T(\delta\gamma, w, \delta x_0) - T(\delta\gamma, w', \delta x_0)\|_{\alpha} \le (\tfrac{L_1}{\alpha-\nu} + \tfrac{L_1\alpha}{1-\alpha\beta})\|\delta\gamma\|_{\alpha}\|\gamma^*(w) - \gamma^*(w')\|_{\beta}.$$

So if we restrict δx_0 to $\|\delta x_0\| \le R$ for any arbitrary $R > 0$, then by the bound estimate (27) we have by Uniform Contraction Principle I

$$\begin{aligned}
\|\delta\gamma^*(w,\delta x_0) - \delta\gamma^*(w',\delta x_0)\|_{\alpha} &\le \tfrac{1}{1-\theta}\|T(\delta\gamma, w, \delta x_0) - T(\delta\gamma, w', \delta x_0)\|_{\alpha} \\
&\le \tfrac{1}{1-\theta}(\tfrac{L_1}{\alpha-\nu} + \tfrac{L_1\alpha}{1-\alpha\beta})\tfrac{\|\delta x_0\|}{1-\theta}\|\gamma^*(w) - \gamma^*(w')\|_{\beta} \\
&\le (\tfrac{L_1}{\alpha-\nu} + \tfrac{L_1\alpha}{1-\alpha\beta})\tfrac{R}{(1-\theta)^2}\tfrac{1}{1-\theta_{cs}}\|w - w'\|,
\end{aligned} \tag{32}$$

where $\delta\gamma = \delta\gamma^*(w,\delta x_0)$, showing $\delta\gamma^*$ is Lipschitz in w for bounded δx_0. Because $\delta p_0(w,\delta x_0)$ is the first point of the sequence $\delta\gamma^*(w,\delta x_0)$, its Lipschitz continuity follows, so is ψ_{cu}'s. □

Lemma 4. *f is a contraction on $\mathcal{F}^s(q)$ uniformly for all $q \in W^{\mathrm{cs}}$.*

Proof. We need to show there is a constant $0 < \varrho < 1$ so that for any $q \in W^{\mathrm{cs}}$ and for any $p, p' \in \mathcal{F}^s(q)$, $\|f(p) - f(p')\| \le \varrho\|p - p'\|$. Let γ_p, $\gamma_{p'}$ be the orbits through p, p', respectively. Then $\delta\gamma^* = \gamma_p - \gamma_q$ and $\delta\gamma^{*\prime} = \gamma_{p'} - \gamma_q$ are fixed points of $T(\cdot, w, \delta x_0)$ and $T(\cdot, w, \delta x_0{}')$, respectively, with $\delta x_0 = p_s - w_s$ and $\delta x_0{}' = p'_s - w_s$. More importantly,

$$\gamma_p - \gamma_{p'} = (\gamma_p - \gamma_q) - (\gamma_{p'} - \gamma_q) = \delta\gamma^*(w,\delta x_0) - \delta\gamma^*(w,\delta x_0{}')$$

whose second point on the sequence is

$$f(p) - f(p') = p_1 - p'_1 = \delta p_1(w,\delta x_0) - \delta p_1(w,\delta x_0{}').$$

The $\mathbb{E}^s$-coordinate of the right side can be estimated as

$$\begin{aligned}
\|\overline{\delta x}_1 - \overline{\delta x}_1{}'\| &\le \|A_s(\delta x_0 - \delta x_0{}') + g_s(q_0(w),\delta x_0) - g_s(q_0(w),\delta x_0{}')\| \\
&\le \nu\|\delta x_0 - \delta x_0{}'\| + \|h_s(q_0(w) + \delta x_0) - h_s(q_0(w) + \delta x_0{}')\| \\
&\le \nu\|\delta x_0 - \delta x_0{}'\| + L\|\delta x_0 - \delta x_0{}'\| \\
&\le (\nu + L)\|p - p'\|
\end{aligned}$$

The $\mathbb{E}^{cu}$-coordinate of the right side is, with $\delta p_i = \delta p_i(w,\delta x_0), \delta p'_i = \delta p_i(w,\delta x_0{}')$,

$$\begin{aligned}
\|\overline{\delta y}_1 - \overline{\delta y}_1{}'\| &\le \textstyle\sum_{i=2}^{\infty} \|A_{cu}{}^{2-i}[g_{cu}(q_i(w),\delta p_i) - g_{cu}(q_i(w),\delta p'_i)]\| \\
&= \textstyle\sum_{i=2}^{\infty} \|A_{cu}{}^{2-i}[h_{cu}(q_i(w) + \delta p_i) - h_{cu}(q_i(w) + \delta p'_i)]\| \\
&\le \textstyle\sum_{i=2}^{\infty} \beta^{i-2} L\alpha^i \|\delta\gamma^*(w,\delta x_0) - \delta\gamma^*(w,\delta x_0{}')\|_{\alpha} \\
&\le \tfrac{L\alpha^2}{1-\alpha\beta}\|\delta\gamma^*(w,\delta x_0) - \delta\gamma^*(w,\delta x_0{}')\|_{\alpha} \\
&\le \tfrac{L\alpha^2}{1-\alpha\beta}\tfrac{1}{1-\theta}\|\delta x_0 - \delta x_0{}'\| \\
&\le \tfrac{L\alpha^2}{1-\alpha\beta}\tfrac{1}{1-\theta}\|p - p'\|
\end{aligned}$$

where (28) is used for the second last estimate. Therefore,

$$\|f(p) - f(p')\| \leq (\nu + L + \tfrac{L\alpha^2}{1-\alpha\beta}\tfrac{1}{1-\theta})\|p - p'\|$$

which implies f is a uniform contraction for sufficiently small L, i.e., for sufficiently small $\|f - Df(\bar{q})\|_1$. □

Lemma 5. *If f is $C^{k,1}$ for $k \geq 1$, then ψ_{cu} is C^k.*

Proof. By the Uniform Contraction Principle II, we need to verify two conditions: (1) $T(\delta\gamma, w, \delta x_0)$ is differentiable in $\delta\gamma$ and $\|D_{\delta\gamma}T(\delta\gamma, w, \delta x_0)\|$ is uniformly bounded by a constant smaller than 1; (2) $T \in C^k(\Delta S_\alpha \times \mathbb{E}^{cs} \times \mathbb{E}^s, \Delta S_\alpha)$.

To show (1), let $\delta\gamma = \{\delta p_n\}, v = \{v_n\} \in \Delta S_\alpha$, and formally differentiate (21). Then $D_{\delta\gamma}T(\delta\gamma, w, \delta x_0)v$ needs to be as below in components:

$$\begin{cases} [D_{\delta\gamma}T(\delta\gamma, w, \delta x_0)v]_{n,\ s} = \sum_{i=1}^{n} {A_s}^{n-i} D_{\delta p}g_s(q_{i-1}(w), \delta p_{i-1})v_{i-1} \\ [D_{\delta\gamma}T(\delta\gamma, w, \delta x_0)v]_{n,\ cu} = \sum_{i=n+1}^{\infty} {A_{cu}}^{n+1-i} D_{\delta p}g_{cu}(q_i(w), \delta p_i)v_i. \end{cases} \tag{33}$$

By the exactly same estimate as for (22) we have

$$\|[D_{\delta\gamma}T(\delta\gamma, w, \delta x_0)v]_{n,\ s}\| \leq \tfrac{L}{\alpha-\nu}\alpha^n\|v\|_\alpha.$$

Similarly, by the exactly same estimate as for (23) we have

$$\|[D_{\delta\gamma}T(\delta\gamma, w, \delta x_0)v]_{n,\ cu}\| \leq \tfrac{L\alpha}{1-\alpha\beta}\alpha^n\|v\|_\alpha.$$

These estimates imply two conclusions. One, because of the uniform convergence of the second equation, it shows the derivative $D_{\delta\gamma}T(\delta\gamma, \delta x_0)$ exists. Two, it shows the derivative is a bounded linear map in $L(\Delta S_\alpha, \Delta S_\alpha)$ whose α-norm

$$\|D_{\delta\gamma}T(\delta\gamma, w, \delta x_0)\|_\alpha \leq \theta(\alpha) < 1,$$

is bounded by the same uniform contraction constant $\theta(\alpha)$ from (24).

To show (2), we separate it into four cases. The first case is for derivatives in x_0, the second case is for derivatives in $\delta\gamma$, the third case is for derivatives in w, and the fourth case is about mixed derivatives. For the first case we note that

$$[D_{\delta x_0}T(\delta\gamma, w, \delta x_0)]_{n,\ s} = {A_s}^n, \text{ and } [D_{\delta x_0}T(\delta\gamma, w, \delta x_0)]_{n,\ cu} = 0.$$

This implies any mixed derivative with δx_0 is the zero operator, hence well-defined and exists. The identity above also shows

$$\|[D_{\delta x_0}T(\delta\gamma, w, \delta x_0)]_n\| \leq \|{A_s}^n\| \leq \alpha^n$$

implying $\|D_{\delta x_0}T(\delta\gamma, w, \delta x_0]\|_\alpha \leq 1$, and $D^j_{\delta x_0}T(\delta\gamma, w, \delta x_0) = 0$, for $2 \leq j \leq k$. Hence, T is C^k in δx_0.

For the second case, the case of $j = 1$ was done above. For any $2 \leq j \leq k$, $[D^j_{\delta\gamma}T(\delta\gamma, w, \delta x_0)]$ needs to be a j-linear form in ΔS_α. To this end, let $v =$

$v^1 \otimes v^2 \otimes \cdots \otimes v^j$ with each $v^\ell \in \Delta S_\alpha,\ 1 \le \ell \le j$. Formally differentiate (21) to get

$$\left\{ \begin{array}{l} [D^j_{\delta\gamma} T(\delta\gamma, w, \delta x_0)v]_{n,\ s} = \sum_{i=1}^n {A_s}^{n-i} D^j_{\delta p} g_s(q_{i-1}(w), \delta p_{i-1}) v_{i-1} \\ {[D^j_{\delta\gamma} T(\delta\gamma, w, \delta x_0)v]}_{n,\ cu} = \sum_{i=n+1}^\infty {A_{cu}}^{n+1-i} D^j_{\delta p} g_{cu}(q_i(w), \delta p_i) v_i, \end{array} \right. \tag{34}$$

where

$$v_i = v_i^1 \otimes v_i^2 \otimes \cdots \otimes v_i^j, \quad v_i^\ell \in \mathbb{R}^d.$$

Similar to the estimate of (22) and because $g_i(q, \delta p) = h_i(q + \delta p) - h_i(q)$, $\alpha < 1$, we have

$$\begin{aligned} \|[D^j_{\delta\gamma} T(\delta\gamma, w, \delta x_0)v]_{n,\ s}\| &\le \textstyle\sum_{i=1}^n \|{A_s}^{n-i}\| \|D^j h_s\| \|v_{i-1}\| \\ &\le \textstyle\sum_{i=1}^n \nu^{n-i} \|h_s\|_j \Pi_{\ell=1}^j \|v_{i-1}^\ell\| \\ &\le \|h_s\|_k \textstyle\sum_{i=1}^n \nu^{n-i} \alpha^{j(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\le \|h_s\|_k \textstyle\sum_{i=1}^n \nu^{n-i} \alpha^{(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\le \tfrac{\|h_s\|_k}{\alpha-\nu} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\alpha. \end{aligned} \tag{35}$$

Similarly, by an exactly same estimate as (23) we can have

$$\begin{aligned} \|[D^j_{\delta\gamma} T(\delta\gamma, w, \delta x_0)v]_{n,\ cu}\| &\le \textstyle\sum_{i=n+1}^\infty \|{A_{cu}}^{n+1-i}\| \|D^j h_{cu}\| \|v_i\| \\ &\le \textstyle\sum_{i=n+1}^\infty \beta^{i-n-1} \|h_{cu}\|_j \alpha^{ji} \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\le \|h_{cu}\|_k \beta^{-n-1} \textstyle\sum_{i=n+1}^\infty (\beta\alpha^j)^i \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\le \|h_{cu}\|_k \beta^{-n-1} \textstyle\sum_{i=n+1}^\infty (\beta\alpha)^i \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\le \tfrac{\|h_{cu}\|_k \alpha}{1-\alpha\beta} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\alpha. \end{aligned} \tag{36}$$

Combine these two estimates to obtain

$$\|[D^j_{\delta\gamma} T(\delta\gamma, w, \delta x_0)]\|_\alpha \le \|(h_s, h_{cu})\|_k \max\{\tfrac{1}{\alpha-\nu}, \tfrac{\alpha}{1-\alpha\beta}\}.$$

The convergence of the infinite series also shows the derivatives are well-defined. This completes the proof that T is C^k in $\delta\gamma$.

For the third case, we will use the property that the fixed point $\delta\gamma^*(w, \delta x_0)$ is also in $\Delta S_{\alpha'}$ for any α' satisfying (26), and the property that the center-stable orbit $\gamma^*(w)$ is a C^k map from $\mathbb{E}^{cs}$ to $S_\mu \subset S_\beta$ for any μ satisfying $\|A_{cs}\| < \mu < \beta$. Specifically, we will take

$$\|A_{cs}\| < \mu = \beta^{1/k} < \beta \ \text{ for } \ k \ge 2,$$

re-adjusting the adapted norm if necessary. In this setting, we will treat T as a composition of a map $\bar{T} : \Delta S_{\alpha'} \times S_\mu \times \mathbb{E}^s \to \Delta S_\alpha$ with the center-stable orbit map $\gamma^* \in C^k(\mathbb{E}^{cs}, S_\mu)$. That is,

$$T(\delta\gamma, w, \delta x_0) = \bar{T}(\delta\gamma, \gamma^*(w), \delta x_0)$$

where $\bar{T}$ is defined by the right side of (12) except for general $\gamma = \{q_n\} \in S_\mu$. Since the center-stable orbit map $\gamma^*(w)$ is C^k, we only need to show $\bar{T}$ is C^k in γ by the chain rule. We will also use the property (10) that

$$\|D^j_q g_i(q, \delta p)\| \le \bar{L} \|\delta p\| \ \text{ with } \bar{L} = \max\{L_j : 1 \le j \le k\}. \tag{37}$$

We are now ready to show $\bar{T}$ is C^k in $\gamma = \{q_n\} \in S_\mu$. We need to show $[D^j_\gamma \bar{T}(\delta\gamma, \gamma, \delta x_0)]$ is a bounded j-linear form from S_μ to ΔS_α. To this end, let $v = v^1 \otimes v^2 \otimes \cdots \otimes v^j$ with each $v^\ell \in S_\mu,\ 1 \le \ell \le j$. Formally differentiate (12) in the general $\gamma \in S_\mu$ to get

$$\begin{cases} [D^j_\gamma \bar{T}(\delta\gamma, \gamma, \delta x_0)v]_{n,\ s} = \sum_{i=1}^{n} {A_s}^{n-i} D^j_q g_s(q_{i-1}, \delta p_{i-1}) v_{i-1} \\ [D^j_\gamma \bar{T}(\delta\gamma, \gamma, \delta x_0)v]_{n,\ cu} = \sum_{i=n+1}^{\infty} {A_{cu}}^{n+1-i} D^j_q g_{cu}(q_i, \delta p_i) v_i, \end{cases} \tag{38}$$

where

$$v_i = v_i^1 \otimes v_i^2 \otimes \cdots \otimes v_i^j, \quad v_i^\ell \in \mathbb{R}^d.$$

Similar to the estimate of (30) and because of (37), $\mu^k = \beta, \alpha'\beta < \alpha$, we have

$$\begin{aligned} \|[D^j_\gamma \bar{T}(\delta\gamma, \gamma, \delta x_0)v]_{n,\ s}\| &\le \sum_{i=1}^{n} \|{A_s}^{n-i}\| L_j \|\delta p_{i-1}\| \|v_{i-1}\| \\ &\le \bar{L} \sum_{i=1}^{n} \nu^{n-i} \alpha'^{i-1} \|\delta\gamma\|_{\alpha'} \Pi_{\ell=1}^{j} \|v_{i-1}^\ell\| \\ &\le \bar{L} \|\delta\gamma\|_\alpha \sum_{i=1}^{n} \nu^{n-i} \alpha'^{i-1} \mu^{j(i-1)} \Pi_{\ell=1}^{j} \|v^\ell\|_\mu \\ &\le \bar{L} \|\delta\gamma\|_\alpha \sum_{i=1}^{n} \nu^{n-i} (\alpha'\beta)^{i-1} \Pi_{\ell=1}^{j} \|v^\ell\|_\mu \\ &\le \bar{L} \|\delta\gamma\|_\alpha \sum_{i=1}^{n} \nu^{n-i} \alpha^{i-1} \Pi_{\ell=1}^{j} \|v^\ell\|_\mu \\ &\le \tfrac{\bar{L}\|\delta\gamma\|_\alpha}{\alpha-\nu} \alpha^n \Pi_{\ell=1}^{j} \|v^\ell\|_\mu. \end{aligned}$$

Similar to the estimate of (31) we have

$$\begin{aligned} \|[D^j_\gamma \bar{T}(\delta\gamma, \gamma, \delta x_0)v]_{n,\ cu}\| &\le \sum_{i=n+1}^{\infty} \|{A_{cu}}^{n+1-i}\| L_j \|\delta p_i\| \|v_i\| \\ &\le \bar{L} \sum_{i=n+1}^{\infty} \beta^{i-n-1} \alpha'^i \|\delta\gamma\|_{\alpha'} \mu^{ji} \Pi_{\ell=1}^{j} \|v^\ell\|_\mu \\ &\le \bar{L} \|\delta\gamma\|_\alpha \sum_{i=n+1}^{\infty} \beta^{i-n-1} (\alpha'\beta)^i \Pi_{\ell=1}^{j} \|v^\ell\|_\mu \\ &\le \bar{L} \|\delta\gamma\|_\alpha \sum_{i=n+1}^{\infty} \beta^{i-n-1} \alpha^i \Pi_{\ell=1}^{j} \|v^\ell\|_\mu \\ &\le \tfrac{\bar{L}\|\delta\gamma\|_\alpha \alpha}{1-\alpha\beta} \alpha^n \Pi_{\ell=1}^{j} \|v^\ell\|_\mu. \end{aligned}$$

Combine these two estimates to obtain

$$\|[D^j_\gamma \bar{T}(\delta\gamma, \gamma, \delta x_0)]\|_\alpha \le (\tfrac{1}{\alpha-\nu} + \tfrac{\alpha}{1-\alpha\beta}) \bar{L} \|\delta\gamma\|_\alpha.$$

The convergence of the infinite series also shows the derivatives are well-defined. Hence $\bar{T}(\delta\gamma, \cdot, \delta x_0)$ is in $C^k(S_\mu, \Delta S_\alpha)$, showing T is C^k in w.

For the fourth case about mixed derivatives of T in all variables, the arguments above for $\delta\gamma, w, \delta x_0$ can be combined to show all derivatives up to order k exist for T. Therefore by the Uniform Contraction Principle II the fixed point $\delta\gamma^*(w, \delta x_0)$ is C^k in both variables. □

Proof of Theorem 1. After the preceding lemmas, it only remains to point out that by the definition of W^{s}, it coincides with the definition of the foliation through the fixed point, $\mathcal{F}^s(\bar{q})$, i.e., $W^{\mathrm{s}} = \mathcal{F}^s(\bar{q})$. In fact, we can show the tangent space directly as below. Since $\bar{q} \sim w = 0$ and $\phi_u(0) = 0$, we have from (20)

$$\psi_{cu}(0, x_0) = \delta y_0(0, x_0) = \sum_{i=1}^{\infty} {A_{cu}}^{1-i} h_{cu}(\delta p_i(0, x_0))$$

whose partial derivative in $x_0 \in \mathbb{E}^s$ at the fixed point $\bar{q} \sim x_0 = 0$ is

$$D_{x_0} \psi_{cu}(0, 0) = \sum_{i=1}^{\infty} {A_{cu}}^{1-i} D h_{cu}(\delta p_i(0, 0)) D_{x_0} \delta p_i(0, 0) = 0$$

since $\delta\gamma^*(0, 0) = \{0\}$, showing $\mathbb{T}_{\bar{q}} \mathcal{F}^s(0) = \mathbb{E}^s$. □

Remark: We can see from the proofs above that if the center-stable manifold point q is fixed at the fixed point $\bar{q}$ throughout, then the extra Lipschitz continuity condition for the highest derivative of f is not needed. That is, the stable manifold $\mathcal{F}^s(\bar{q}) = W^{\mathrm{s}}$ is C^k if f is C^k. This is because in this case, $g(\bar{q}, \delta p) = h(p)$ with $\bar{q} = 0,\ \delta p = p$.