8. Chaotic Orbit

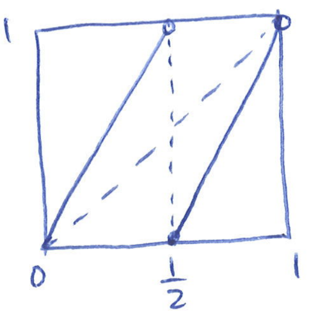
Definition: Let $f: I \to I$ and let $\gamma(x_0) = \{x_0, x_1, \cdots\}$ be a bounded orbit. The orbit is chaotic if

(1) It is not asymptotically periodic.

(2) It has either sensitive dependence on initial condition or the Lyapunov exponent $h(x_0) > 0$.

Example: Every orbit of the tent map $f(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$ that is not eventually periodic is a chaotic orbit. In fact, by using the itinerary, every asymptotically periodic point is ~~in fa~~ eventually periodic. As we worked out ~~an~~ in an example earlier, the Lyapunov exponent of any non-eventually periodic orbit is $\ln 2 > 0$. So by definition, every non-eventually periodic orbit is chaotic.

Example: Consider the map $f(x) = 2x \pmod 1$, see the graph.



Let $L = [0, \frac{1}{2})$, $R = [\frac{1}{2}, 1)$. Then $K_f = \overline{RL}$.
One can show that the binary expansion of any number $x \in [0, 1)$ which replaces any tail sequen of the form $0111\cdots$ by $1000\cdots$ coincides with the itinerary expansion. That is, let $S(x_0) = S_0 S_1 \cdots$, then $x_0 = \frac{b_0}{2} + \frac{b_1}{2^2} + \frac{b_2}{2^3} + \cdots$ identifying $b_i = 0$ iff $S_i = L$.
For example, the number $\frac{1}{2} = .1000\cdots$ in its binary expansion and $S(\frac{1}{2}) = RLLL\cdots$ in its itinerary expansion. Also note $f(x_0) = 2(\frac{b_0}{2} + \frac{b_1}{2^2} + \cdots) \pmod 1 = \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots = .b_1 b_2 \cdots$. Like the tent map, all non-eventually periodic orbits are chaotic.

Hwk. ~~The~~ consider the map $f(x) = 2x^2 - 5x$ on $\mathbb{R}$ has fixed point at $x=0$ ~~and $x=3$,~~ (a) Find all the fixed points

(b) Determine the stability of the fixed points.

(c) Find all period-2 points in $[0, 2.5]$ (Use Maple)

(d) Determine the stability of the period-2 points.

Hwk (a) Find a period-3 point for the tent map $T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$ that satisfies $a < \frac{1}{2}$, $T(a) < \frac{1}{2}$, $T^2(a) > \frac{1}{2}$ and $T^3(a) = a$.

(b) Show that all periodic points are repellers.

Hwk: Show that if $x_n = f(x_{n-1})$, $\lim_{n\to\infty} x_n = \bar{x}$ exists and f is continuous at $\bar{x}$, than $\bar{x}$ must be a fixed point of f.

8 Natural Measure

It is impossible to predict where a chaotic orbit will be at a particular iterate into the future. But it is possible to anticipate what it will likely do in some probabilistic sense. The idea of natural measure is a way to specifying the likely-hood a typical orbit will visit any conceivable region. To do this, we iterate a map f at a randomly chosen initial point x_0, count the amount of iterates in a particular region, say S, divide by the number of iterates, and take the limit as the number of iterates goes to infinity. This gives rise to the so-called "fraction" of iterates of the orbit $\{f^n(x_0)\}$ lying in the set S. We denote it by

$$\Psi_f(x_0, S) = \lim_{n \to \infty} \frac{\#\{f^i(x_0) \in S : 0 \leq i \leq n\}}{n}$$

if the limit exists. The concept of natural measure is a slight modification on the "fraction of iterates". More precisely, let $N_r(S)$ be the set of points that are within $r>0$ distance to the set S, called the r-neighborhood of S. The "natural measure" generated by f is defined by

$$\mu_f(S) = \lim_{r \to 0} \Psi_f(x_0, N_r(S))$$

for each closed set S, if the limit exists for almost all $x_0 \in I$.

Notice that if the natural measure exists, it must be a probability measure, meaning $\mu(I) = 1$. Also, it must be f-invariant in the sense that

$$\mu(A) = \mu(f^{-1}(A)),$$

which simply ~~states that~~ is based on the fact that the number of iterates of $\{f^i(x_0): 0 \le i \le n\}$ that visit a neighborhood of A must be the same number of iterates that come from or have already visited the preimage $f^{-1}(A)$.

We will consider in this section only those maps $f: [0,1] \to [0,1]$ which Markov in the following sense

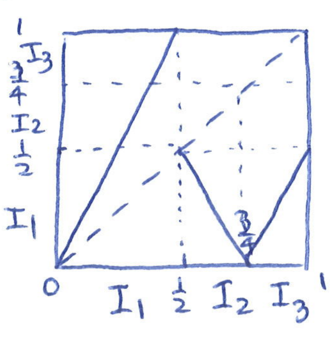
(a) There is a partition of the inter $[0,1]$ into finite union of subintervals I_i, $i = 1, 2, \cdots, n$. That is

$$[0,1] = I_1 \cup I_2 \cup \cdots \cup I_n$$

(b) f is linear on each of the subinterval I_i.

(c) The image $f(I_i)$ of each I_i is a union of some subintervals of I_j, i.e. if $f(I_i) \cap I_j \neq \emptyset$, then $f(I_i) \supset I_j$

Example:



The map f shown has a Markov partition $[0,1] = I_1 \cup I_2 \cup I_3$ with $I_1 = [0, \frac{1}{2}]$, $I_2 = (\frac{1}{2}, \frac{3}{4}]$, $I_3 = (\frac{3}{4}, 1]$. f is linear on each I_i with slope $|f'(x)| = 2$. Also, $f(I_1) = I_1 \cup I_2 \cup I_3 = [0,1]$, $f(I_2) = f(I_3) = I_1$. So f is markov.

Example 2: Both the tent map $T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$ and the doubling map $D(x) = 2x \pmod 1$ are Markov as well.

The question for a Markov map would be what would ~~be~~ the natural measure, if exists, of each of the partitioning interval I_i? To answer this question and the existence question, we need to introduce the concept of "transition matrix". Let $[0,1] = \bigcup_{i=1}^{n} I_i$ be a Markov partition of a Markov map f. Then for any $1 \le i \le n$ and $1 \le j \le n$, the preimage $f^{-1}(I_j)$ either intersects I_i empty or a ~~non-em~~ subinterval of I_i. Either ways we denote ~~the~~ $f^{-1}(I_j) \cap I_i$ the subinterval (empty interval allowed) in I_i. Let $|I_i|$ denote the length of interval I_i ~~Then~~ and introduce the ratio

$$p_{ij} = \frac{|f^{-1}(I_j) \cap I_i|}{|I_i|}$$

which can be regarded as the probability of $f(x)$ being in I_j given that x is I_i. Then by definition, the "transition matrix" P of f is the $n \times n$ matrix with entries p_{ij}, i.e. $P = [p_{ij}]_{n \times n}$.

Example 3. The transition matrix P for the map from example 1 above is $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

<u>Example 4</u>: the transition matrix P for both the tent map and the doubling map is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ with the Markov partition $[0,1] = [0,\frac{1}{2}) \cup [\frac{1}{2},1]$.

<u>Definition</u>: A transition matrix P is regular if for some K, P^K has no zero entries.

<u>Example 5</u>: The transition matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is regular by definition with $K=1$. The transition matrix $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is regular with $K = 2$ because $P^2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

The question of existence of natural measure is answered by the following theorem which we will ~~not prov~~ omit the proof.

<u>Theorem</u>: If f is Markov and its transition matrix is regular, then the natural measure exists for f. Moreover, the natural measure on each of the partitioning subinterval I_i has a constant density ρ_i, that is if $J \subset I_i$ is a subinterval of I_i, then the natural measure of J is calculated as $\mu(I) = \rho_i |J|$ with $|J|$ being the length of J. $\square$

For the remainder of this section, we will illustrate a method to calculate the natural measure of partitioning intervals I_i.

Here is how we derive the method. Because μ is f-invariant, $\mu(J)=\mu(f^{-1}(J))$. We also have $\mu(I_i) = q_i|I_i|$ with $q_i>0$ being the density. We will also use the fact that if $A_1, A_2, \cdots, A_k$ are pair-wise disjoint, i.e, no two different sets intersect ~~a~~ nonempty interval, then $\mu(\bigcup_{i=1}^{k} A_i) = \sum_{i=1}^{k}\mu(A_i)$. We now ready to calculate $\mu(I_j)$.

$$\mu(I_j) = \mu(f^{-1}(I_j)) = \mu(f^{-1}(I_j)\cap[0,1]) = \mu\Big(f^{-1}(I_j)\cap\big(\bigcup_{i=1}^{n} I_i\big)\Big)$$

$$= \mu\Big(\bigcup_{i=1}^{n} f^{-1}(I_j)\cap I_i\Big) = \sum_{i=1}^{n}\mu(f^{-1}(I_j)\cap I_i)$$

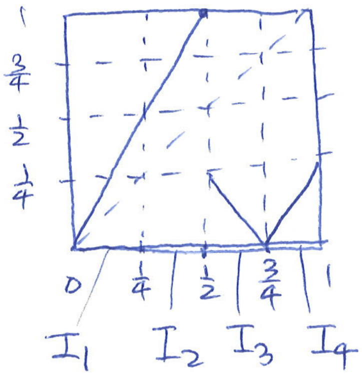
$$= \sum_{i=1}^{n} q_i\,|f^{-1}(I_j)\cap I_i| = \sum_{i=1}^{n} q_i \frac{|f^{-1}(I_j)\cap I_i|}{|I_i|}\,|I_i|$$

$$= \sum_{i=1}^{n}\mu(I_i)\,p_{ij}, \text{ with } p_{ij} \text{ the transition matrix entry}$$

This formulan says that as a vector $\vec{u} = (\mu(I_1), \mu(I_2), \cdots, \mu(I_n))$, $\vec{u}$ is an eigenvector for transpose of the transition matrix, i.e. $P^T\vec{u} = \vec{u}$, and the associated eigenvalue is 1.

<u>Example 6</u>: Let $P=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$. Solving $P^T\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}u_1\\u_2\end{pmatrix}$ gives $u_1=u_2$. Since μ is a probabilistic measure, we need $u_1+u_2=1$ as well. So the only solution to $u_1=u_2$, $u_1+u_2=1$ is $u_1=u_2=\frac{1}{2}$. In other words, for the tent map, any typical orbit has an equal probability to visit both subinterval $[0,\frac{1}{2})$, $(\frac{1}{2},1]$. For $P=\begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{4}\\ 1 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}$, $P^T=\begin{bmatrix}\frac{1}{2} & 1 & 1\\ \frac{1}{4} & 0 & 0\\ \frac{1}{4} & 0 & 0\end{bmatrix}$, and solving $P^T\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ gives $\frac{1}{4}u_1=u_2$, $\frac{1}{4}u_1=u_3$. Using $u_1+u_2+u_3=1$ also gives $u_1=\frac{2}{3}$, $u_2=u_3=\frac{1}{6}$. In terms of density, $q_1=\frac{4}{3}$, $q_2=q_3=\frac{2}{3}$ for the map of example 1.

Example: Find the Lyapunov number for the map as shown.



Soln: We find first the natural measures on the Markov intervals.

$I_1=[0,\frac{1}{4}),\ I_2=[\frac{1}{4},\frac{1}{2}),\ I_3=[\frac{1}{2},\frac{3}{4}),\ I_4=[\frac{3}{4},1].$

In fact, f is Markov because f is linear on each I_i, and

$f(I_1)=I_1\cup I_2,\ f(I_2)=I_3\cup I_4,\ f(I_3)=f(I_4)=I_1.$

So the transition matrix is
$$P=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$
One can verify explicitly that P is regular because P^4 has no zero entries. Therefore, the natural measure exists. Moreover let u_1, u_2, u_3, u_4 be the measures on I_1, I_2, I_3, I_4, then $\vec{u}=(u_1,u_2,u_3,u_4)$ satisfies $\vec{u}=P^T\vec{u}$ and $u_1+u_2+u_3+u_4=1$.
Solve $\vec{u}=P^T\vec{u}$, we have $u_2=\frac{1}{2}u_1,\ u_3=\frac{1}{2}u_2=\frac{1}{4}u_1,\ u_4=\frac{1}{2}u_2=\frac{1}{4}u_1$.
Use $u_1+u_2+u_3+u_4=1$ to get $u_1+\frac{1}{2}u_1+\frac{1}{4}u_1+\frac{1}{4}u_1=2u_1=1$, so $u_1=\frac{1}{2}$ and $u_2=\frac{1}{4},\ u_3=\frac{1}{8},\ u_4=\frac{1}{8}$. We are no ready to compute the Lyapunov ~~exponent~~ number. We note first that $|f'(x_i)|=2$ if $x_i\in I_1\cup I_2$, $|f'(x_i)|=1$ if $x_i\in I_3\cup I_4$. So for all most all initial point $x_0\in(0,1)$, we have the following

$$L(x_0)=\lim_{n\to\infty}\left(\prod_{i=0}^{n}|f'(x_i)|\right)^{\frac{1}{n}}=\lim_{n\to\infty}\Big(\prod_{x_i\in I_1\cup I_2}|f'(x_i)|\Big)^{\frac{1}{n}}\Big(\prod_{x_i\in I_3\cup I_4}|f'(x_i)|\Big)^{\frac{1}{n}}$$
$$=\lim_{n\to\infty}\left(2^{\#\{x_i\in I_1\cup I_2;\ 0\le i\le n\}}\right)^{\frac{1}{n}}\left(1^{\#\{x_i\in I_3\cup I_4;\ 0\le i\le n\}}\right)^{\frac{1}{n}}$$
$$=2^{\lim_{n\to\infty}\frac{\#\{x_i\in I_1\cup I_2;\ 0\le i\le n\}}{n}}=2^{u_1+u_2}=2^{\frac{1}{2}+\frac{1}{4}}=\boxed{2^{\frac{3}{4}}}>1\ \square.$$

(Note: density: $\rho_1=\frac{1}{2}/\frac{1}{4}=2,\ \rho_2=\frac{1}{4}/\frac{1}{4}=1,\ \rho_3=\rho_4=\frac{1}{8}/\frac{1}{4}=\frac{1}{2}$, i.e. $\gamma(x_0)$ in I_1 is twice as dense as in I_2 which $\gamma(x_0)$ is twice as dense as in I_3 and I_4.)

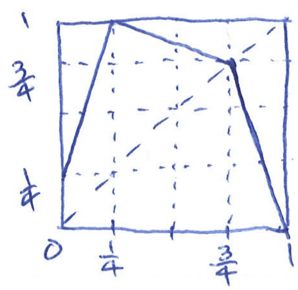
Take Home Final Math 442

(10) 1. Find extremals for the functional

$$\int_a^b (y^2+y'^2+2ye^x)\,dx\,,\quad y\in C^2,\ y(a)=y_a\,,\ y(b)=y_b$$

(10) 2. Show that the functional $J(y)=\int_a^b (p(x)y'^2+q(x)y^2)\,dx$, $y\in C^2$, $y(a)=y_a$, $y(b)=y_b$, where p and q are positive, is an absolute minimum for $y=Y(x)$, where Y is the solution to the Euler equation. (Hint: Show that $\varepsilon=0$ is an absolute minimum for $K(\varepsilon)=J(Y+\varepsilon h)$ for every admissible variation h.)

(15) 3. Consider the piecewise linear map f as shown.



a) Show that the fixed point $x=\frac{3}{4}$ is a repeller by showing that for every $x_0\neq\frac{3}{4}$ sufficiently near $\frac{3}{4}$, $f^i(x_0)$ moves away from $\frac{3}{4}$.

b) Find a period-2 point in $[0,\frac{1}{4})$

c) Find a period-3 point in $[0,\frac{1}{4})$

(15) 4. Consider the same map f as in 3 above. Let $L=[0,\frac{1}{4})$, $M=[\frac{1}{4},\frac{3}{4})$, $R=[\frac{3}{4},1]$. Let $S(x_0)=S_0S_1S_2\cdots$ be the itinerary of a point $x_0\in[0,1]$ with $S_i\in\{L,M,R\}$.

a) Show that $S(x_0)$ must satisfy the following rules (see the so-called transition graph)

$$\begin{tikzcd} L \arrow[r] \arrow[dr, shift right] & M \arrow[d, shift right] \\ & R \arrow[ul, shift right] \arrow[u, shift right] \end{tikzcd}$$

i) If $S_i=L$ then $S_{i+1}\neq L$.

ii) If $S_i=M$ then $S_{i+1}\neq L, M$.

iii) If $S_i=R$ then $S_{i+1}\neq R$ or if $S_{i+1}=R$ then $S_k=R$ for all $k\geq i$

either $S_k=R$ for all $k\geq i$ or $S_{i+1}\neq R$

b) Assume that any sequence $S=S_0S_1S_2\cdots$ obeying the rules a) i), a) ii), a) iii) above is the itinerary of a unique point x in $[0,1]$, then construct a chaotic orbit by showing its Lyapunov number is greater than 1.

(20) 5. Consider the same map as in 4 above.

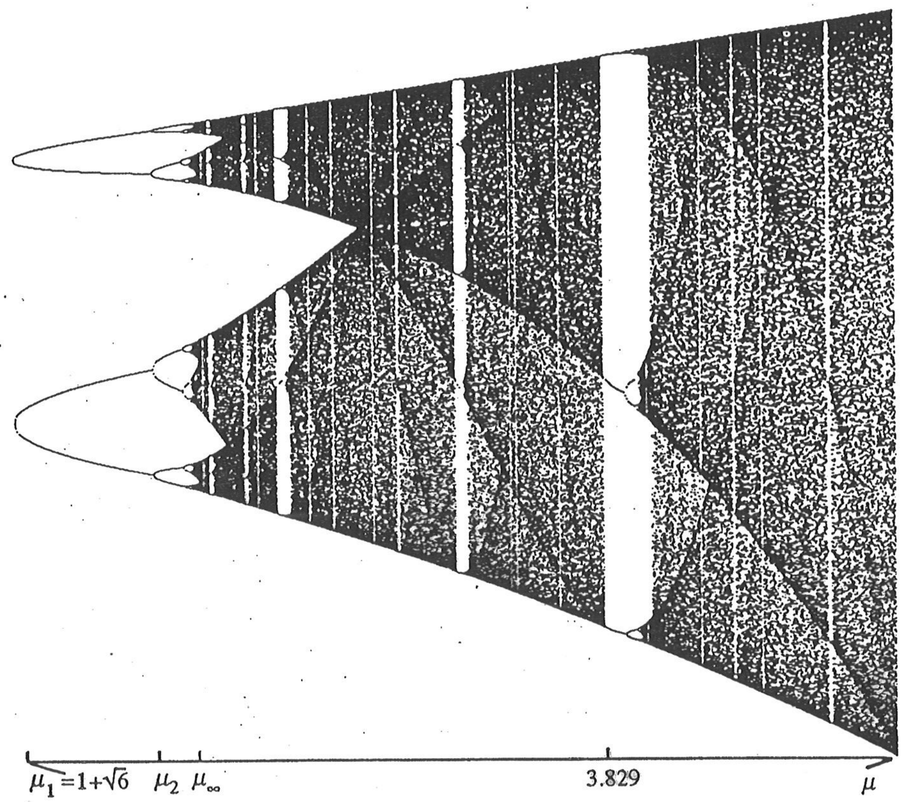
a) Show that f is Markov (you need to find a Markov partition)

b) Show the transition matrix is regular.

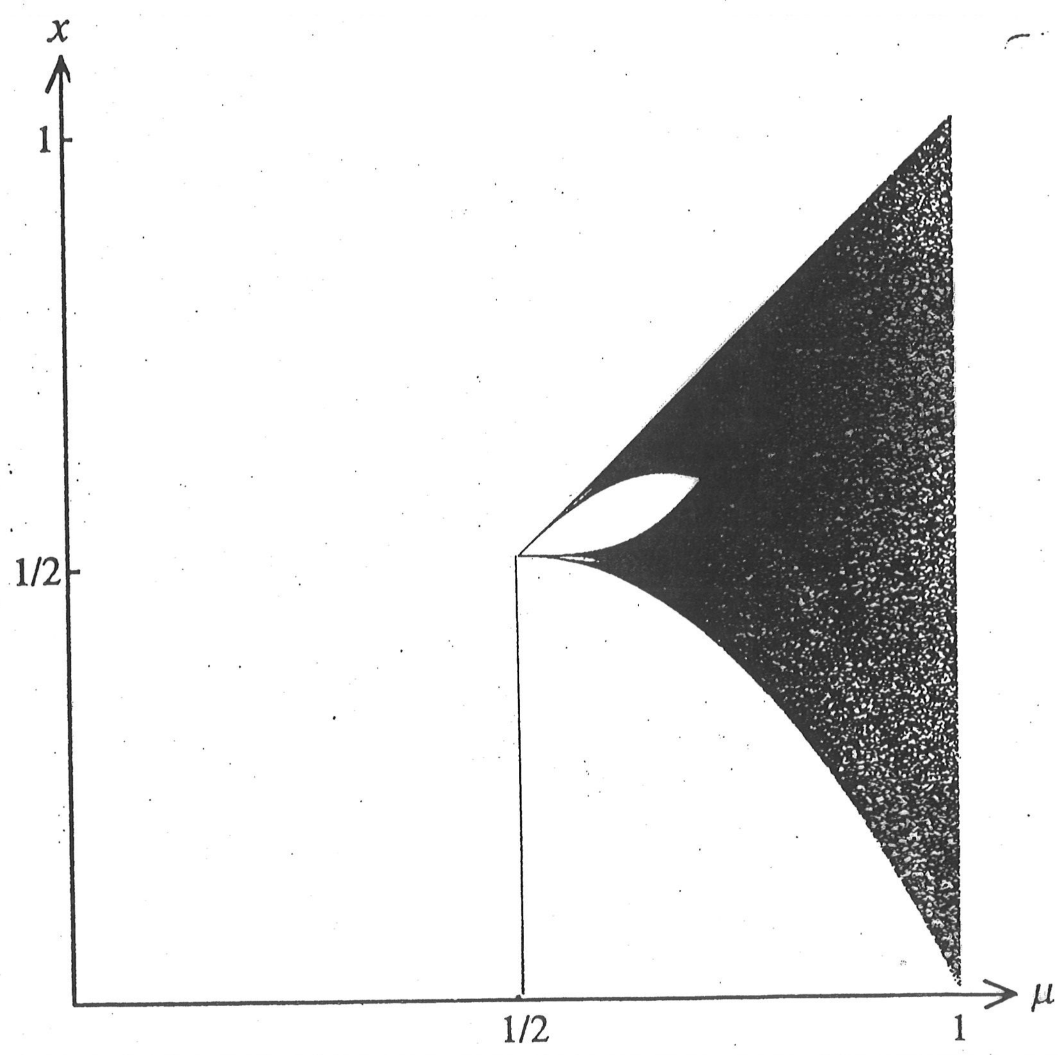
c) Find the natural measures and their densities on the Markov Partition.

d) Find the Lyapunov number for any typical orbit. $\square$

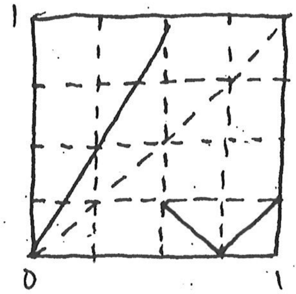
e) Numerically verify the answers of d)



remainder of the bifurcation diagram for $\{Q_\mu\}$

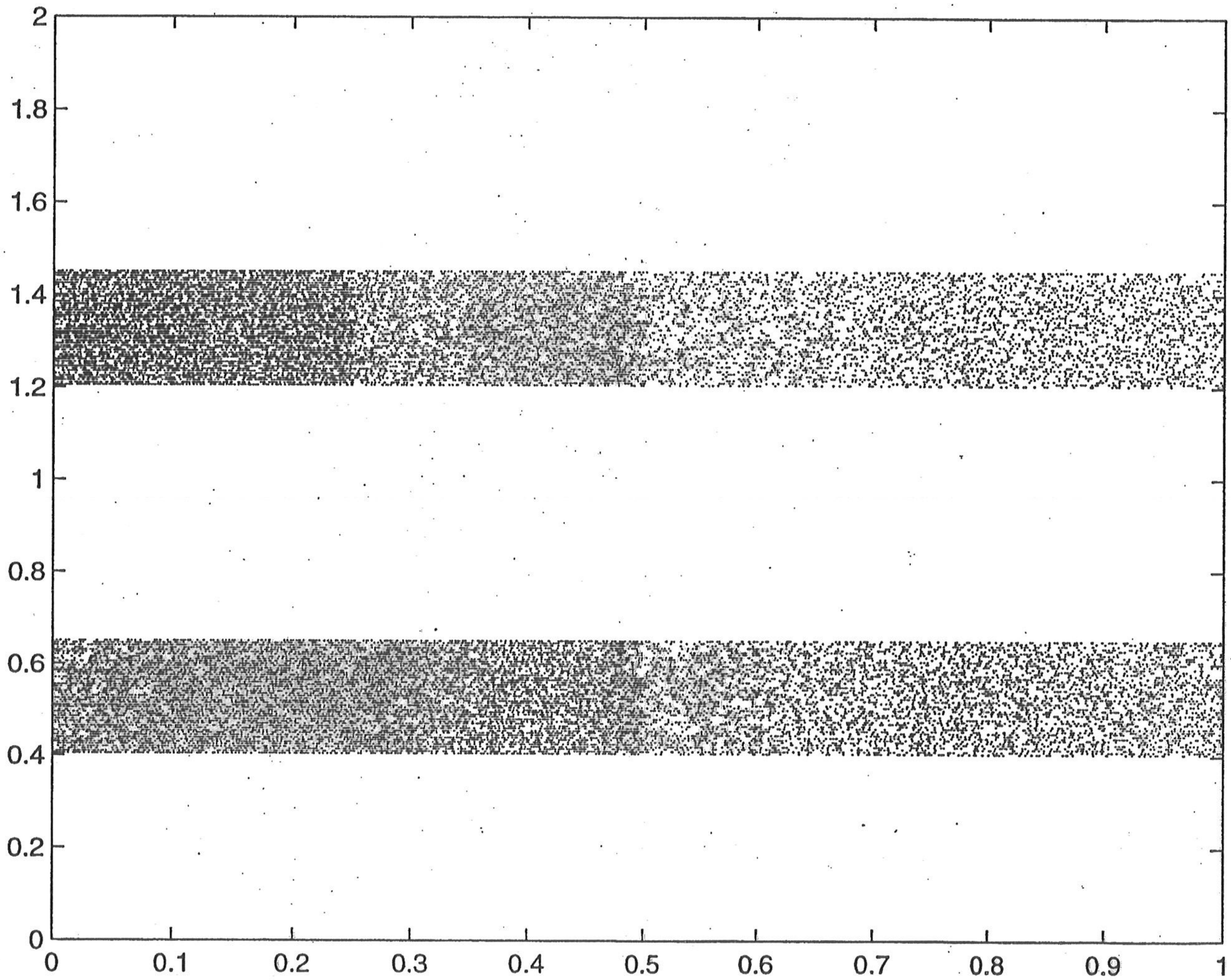


Top Graph

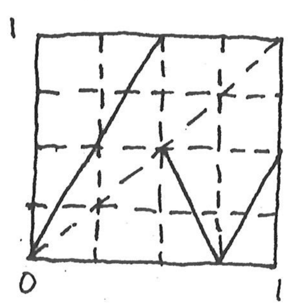


$$\rho = \begin{cases} 2, & 0 \le x \le \frac{1}{4} \\ 1, & \frac{1}{4} \le x \le \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \le x \le \frac{3}{4} \\ \frac{1}{2}, & \frac{3}{4} \le x \le 1 \end{cases}$$

(Natural measure density)

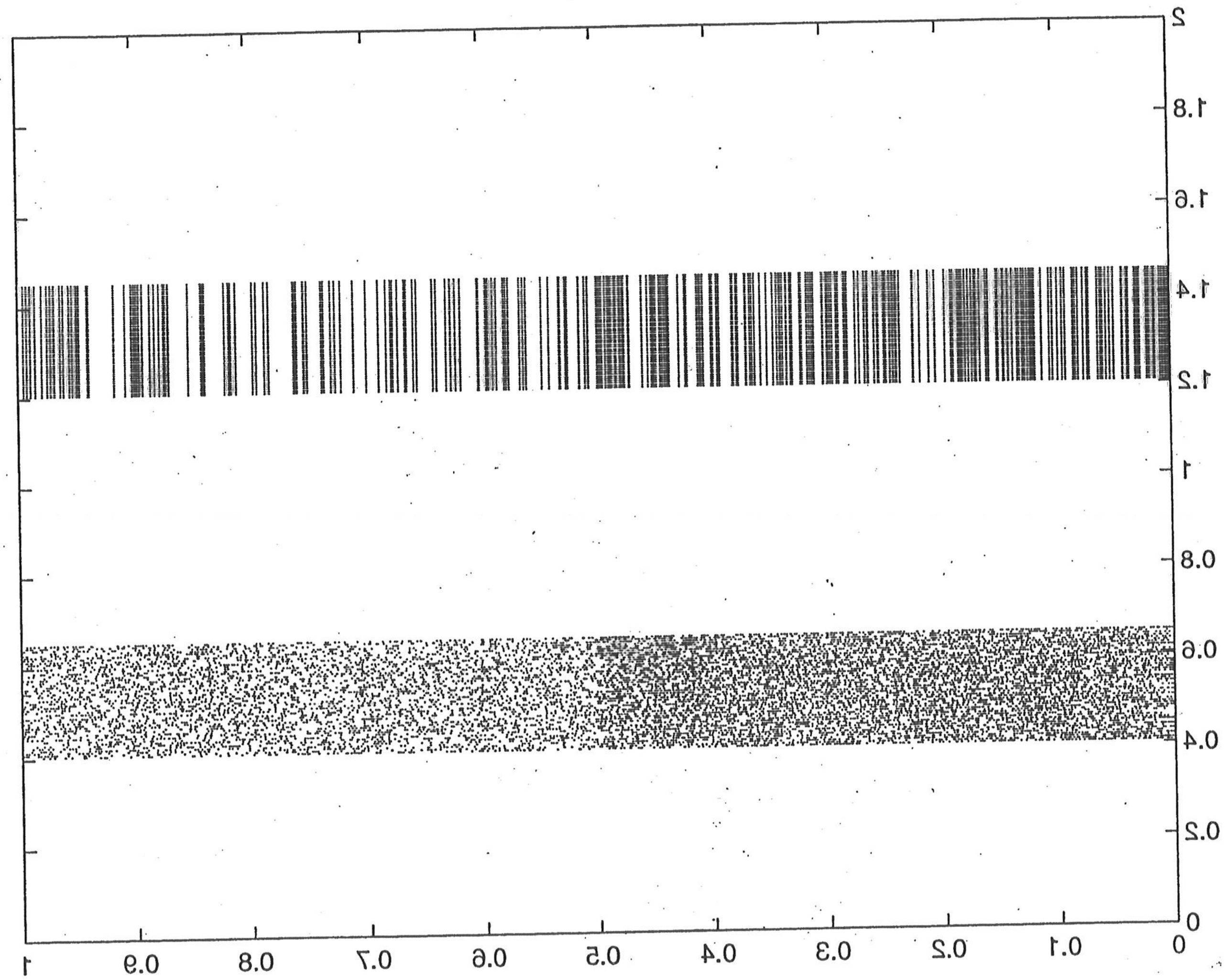


Bottom Graph



$$\rho = \begin{cases} \frac{4}{3}, & 0 \le x \le \frac{1}{2} \\ \frac{2}{3}, & \frac{1}{2} \le x \le \frac{3}{4} \\ \frac{2}{3}, & \frac{3}{4} \le x \le 1 \end{cases}$$

(natural measure density)



0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

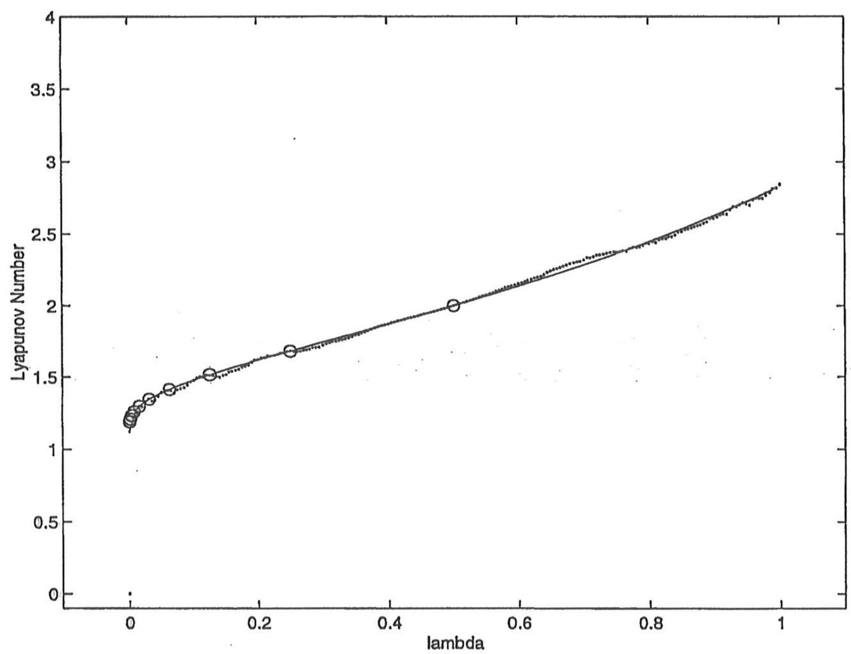


Figure 14: Approximate Lyapunov numbers taken over 10,000 iterates compared to the Lyapunov numbers calculated in Section 5.