4. Introduction to Chaos

1. Solutions to Ordinary Differential Equations.

Consider the initial value problem of an autonomous ordinary differential equations

$$\dot{x} = f(x), \; x(0) = x_0, \quad x \in \mathbb{R}^n, \; f : \mathbb{R}^n \to \mathbb{R}^n.$$

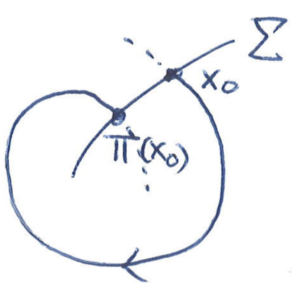
Denote the solution by $x(t) = x(t, x_0)$, or alternatively $x(t) = \phi^t x_0$. By a fundamental result of differential equations, we know that if the function f is of class C^r, $r \geq 1$, then the solution $x(t, x_0)$ exists; it is unique satisfying the initial condition $x(0, x_0) = x_0$; and the function $x(t, x_0)$ in both t and x_0 is also C^r, i.e. as smooth as the right hand side f.

As an important consequence to the uniqueness, to solutions $x(t, x_0)$, $x(t, y_0)$ are either identical $x(t, x_0) \equiv x(t, y_0)$, in which case $x_0 = y_0$ or never cross each other, i.e, $x(t, x_0) \neq x(t, y_0)$ for any t, in which case $x_0 \neq y_0$. Therefore, in the so-called phase space $\mathbb{R}^n$, there is exactly one solution curve going through one point and the space is filled with solution curves, called either trajectories or orbits.

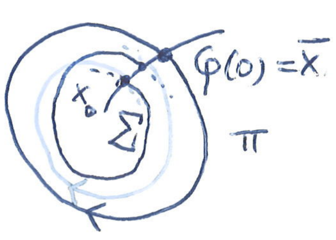
The simplest trajectories or orbits are the steady states x_0 at which $f(x_0) = 0$. So $x(t) \equiv x_0$ is the constant solutions. The next simplest orbits are the periodic orbits $x(t, x_0)$ which satisfies $x(T+t, x_0) = x(t, x_0)$ for all t and some $T > 0$.

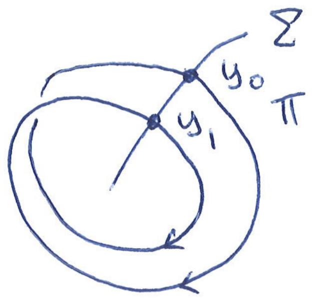
2. Poincaré Return Maps.

Let Σ be a hyperplane in $\mathbb{R}^n$ with the property that at each point p of Σ, the vector field $f(p)$ points away from Σ. This property is referred to as that Σ is transversal to the vector field f, and such a hyperplane is called a Poincaré cross-section. Any trajectory- or flow-induced map, if possible, from Σ to Σ is called a Poincaré return map. Denoted by $\pi : \Sigma \to \Sigma$. It satisfies $\pi(x_0) = x(\tau(x_0), x_0)$ for those $x_0 \in \Sigma$ for which there exist the 1st return time $t = \tau(x_0)$ so that $x(t, x_0) \notin \Sigma$, $0 < t < \tau(x_0)$, but $x(\tau(x_0), x_0) \in \Sigma$. Note that as with any mapping, the domain of π may not be all Σ. It may even be an empty set if none of the point of Σ returns.



The key advantage of studying Poincaré maps is dimension reduction! For example, if $\varphi(t)$ is a periodic orbit with period $T > 0$, then the Poincaré map, when properly defined as above, will have a corresponding fixed point $\varphi(0)$, i.e., $\pi(\bar{x}) = \bar{x}$, with $\bar{x} = \varphi(0)$. If there is another periodic orbit $\psi(t)$ nearby, which takes roughly $2T$ unit of time to return, then $y_0 = \psi(0)$ is a period-2 point of π, i.e. $\pi(y_0) = y_1$, $\pi(y_1) = y_0$, $y_1 \neq y_0$.





Not only such orbits are reduced to periodic points of map, but also ~~th~~ the nature of these orbits is also captured by the map. For example, if the periodic orbit $\varphi(t)$ attracts every other orbit nearby, i.e., $x(t, x_0) \to \varphi$ distance-wise as $t \to +\infty$, then the sequence of returning points $\{x_n\}$, $x_n = \Pi(x_{n-1})$, $n = 1, 2, \cdots$, converges to $\bar{x} = \varphi(0)$. Such periodic orbit φ is called stable and the corresponding fixed point $\bar{x}$ of Π is also called stable.

In conclusion, the dynamics of the equation $\dot{x} = f(x)$ can be uniquely translated into the dynamics of a Poincaré return map, if such a map indeed exists, and the map is 1-dimensional less than the phase space.

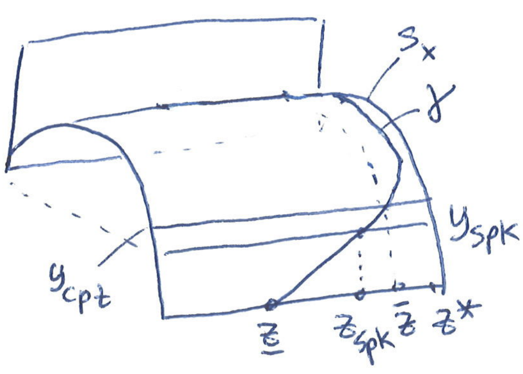
3. Return Maps in the Food Chain Model.

Consider the dimensionless food chain model

$$\begin{cases} \zeta \dot{x} = x\left(1 - x - \dfrac{y}{\beta_1 + x}\right) \\ \dot{y} = y\left(\dfrac{x}{\beta_1 + x} - d_1 - \dfrac{z}{\beta_2 + y}\right), \quad 0 < \zeta \ll 1, \; 0 < \varepsilon \ll 1 \\ \dot{z} = \varepsilon z\left(\dfrac{y}{\beta_2 + y} - d_2\right) \end{cases}$$

at the singular limit $\zeta = 0$. Refer to previous lecture notes for definitions of the ~~following~~ expressions and quantities that follow. Consider the configurations

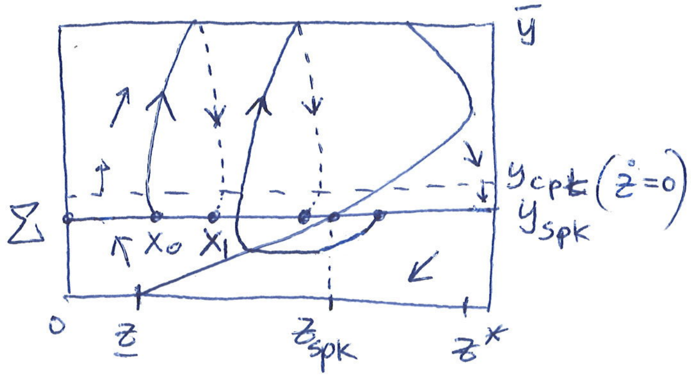
$$\underline{z} < z_{spk} < \bar{z} < z^*$$



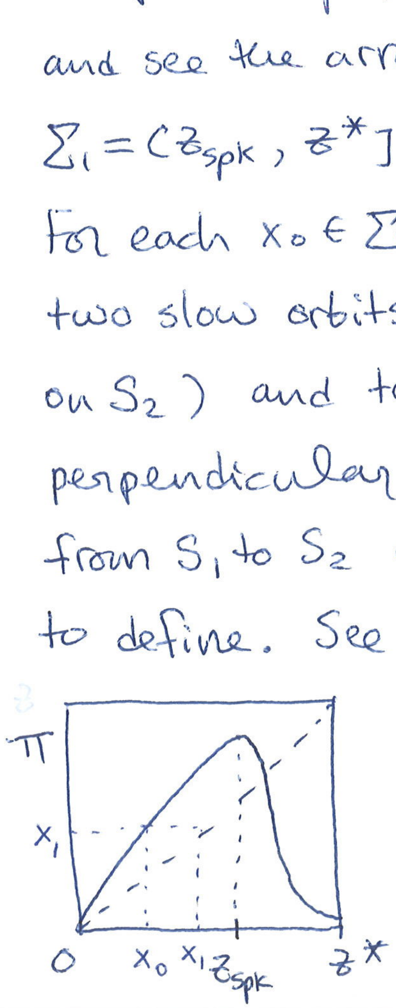
and

$$y_{spk} < y_{cpt}$$

(We considered the other case $y_{cpt} < y_{spk}$ previously).



As we argued before, we project the slow flow structure onto the yz-plane as shown and use it to piece together the limiting slow and fast solutions in concatenation. This reduction to 2-dimensional slow manifold S_1, S_2 is possible with $\varsigma = 0$ and for $\varsigma > 0$ it provides a good approximation. The point here is that this reduction can be further reduced to a 1-dimensional Poincaré map! In fact, let $\Sigma = \{y = y_{spk}\}$ be the cross-section. The reduced slow equation in y and z is transversal to Σ everywhere except at (y_{spk}, z_{spk}) at which $\dot{y} = 0$. (Use phase plain analysis and see the arrows.) The two intervals $\Sigma_0 = [0, z_{spk})$, $\Sigma_1 = (z_{spk}, z^*]$ are transveral to the (y,z)-vector field. For each $x_0 \in \Sigma$, the return map $\pi(x_0)$ takes exactly two slow orbits (one solid curve on S_1, one dotted curve on S_2) and two fast orbits (not shown for being perpendicular to the (y,z)-plane, one jumps at $y = \bar{y}$ from S_1 to S_2 and the other jumps at $y = y_{spk}$ from S_2 to S_1) to define. See the illustration with $\pi(x_0) = x_1$. Notice the following properties (see the graph of π):



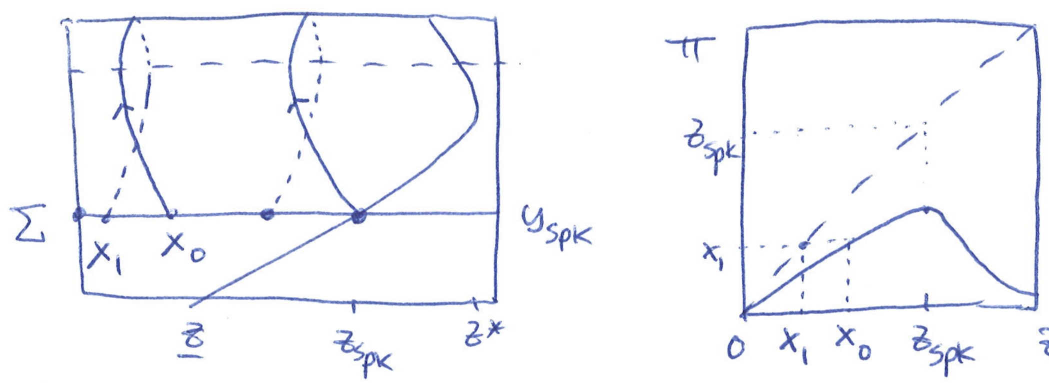
ⓐ $\pi(0) = 0$, corresponding a limiting relaxation oscillation.

ⓑ $\pi(z_{spk})$ is the maximum value.

ⓒ π is increasing on Σ_0 and decreasing on Σ_1.

ⓓ One can show that π is C^1 and $\pi'(z_{spk}) = 0$.

(Food chain return map continued). As one increases the parameter δ_2, the z-nullcline $\{y = y_{cpt}\}$ moves upward. The illustrations contained in the last page are valid for y_{cpt} slightly above y_{spk}. As the gap $y_{cpt} - y_{spk} > 0$ increases, the graph of the return map π should drop as depicted in the illustration below. Because in such



a case, the leftward movement of the trajectory overtakes its rightward movement, the returning point $x_1 = \pi(x_0) < x_0$ is ~~to~~ smaller than where it starts. As we will explain later in the class, this change in the profile of the return map as a parameter changes will give rise to the apperance of periodic ~~and~~ orbits of period $1, 2, 4, 8, \cdots, 2^n, \cdots$, the so-called "period-doubling cascade", a signiture of chaotic dynamics in many problems in applied mathematics.

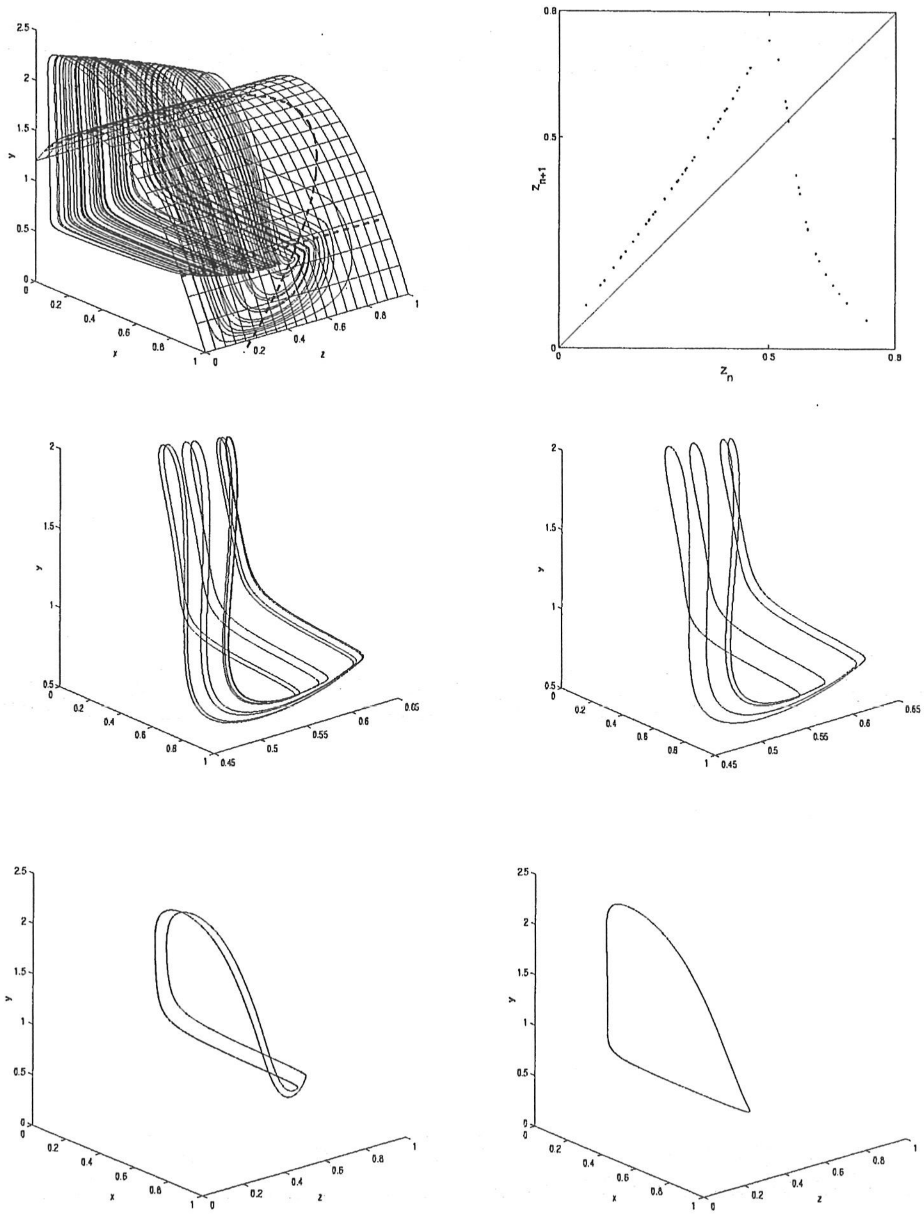


FIGURE 3. A chaotic attractor for $\delta_2 = 0.62$ in the top-left figure. The curve γ and the z-nullcline on $\mathcal{S}_1^x \cup \mathcal{S}_3^x$ are the dashed curves. The top-right figure is the z-projection of the Poincaré return map which is shown to be close to the 1-dimensional, limiting Poincaré map. The remaining figures show that the system goes through a reversed period-doubling cascade for parameter values $\delta_2 = 0.67785, 0.67823, 0.68, 0.7155$ for period 8, 4, 2, 1 orbits respectively.

§2. One-Dimensional Maps.

(Cobweb Plot)

1. By transform any interval $[a,b]$ to $[0,1]$ by a linear map, we can always assume a map defined from $[a,b]$ into $[a,b]$ to be a map from $[0,1]$ to $[0,1]$. Therefore, throughout, we will consider on 1-dimensional maps

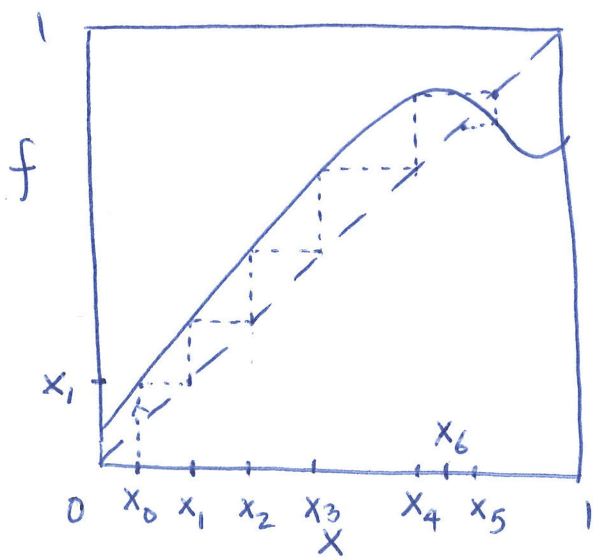
$$f:[0,1]\to[0,1].$$

Definitions: ① The orbit of a point $x\in[0,1]$ under f is the set $\gamma(x):=\{x, f(x), f^2(x), \cdots, f^n(x), \cdots\}$, where $f^n(x) = f(f^{n-1}(x))$, called the nth iterate of x. The starting point x is called the initial point. Often, we use $x_n=f^n(x)$, $x_0=x$.

② A point p is called a fixed point if $f(p)=p$.

③ A point p is called a period-k point for $k\ge 1$ if $f^i(p)\ne p$ for $0<i<k$ and $f^k(p)=p$. Thus, the orbit contains only k distinct points.

Any orbit can be graphically represented by the so-called cobweb plot, making use of the diagonal line $y=x$ as a reference device. See the illustration. Here is how it works.



Start at any initial x_0, go vertically to the graph of f to locate (x_0,x_1). Go horizontal to locate the point (x_1,x_1) on the diagonal $y=x$, and then repeat the steps to locate (x_1,x_2), (x_2,x_2), and so on. The x-coordinates of these points give you the orbit $\gamma(x_0)=\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$.

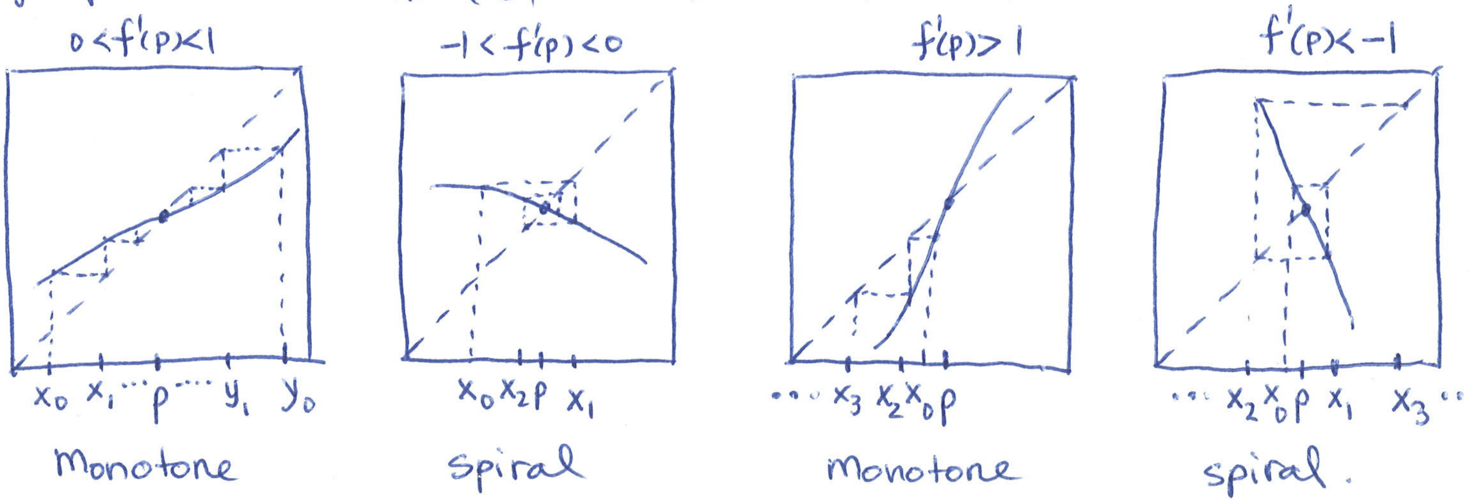
2. Stability of Fixed Points and Periodic Points.

Definition: A fixed point p is called a sink or attracting fixed point if the ~~orbit of~~ iterates of all points sufficiently close to p are attracted to p. More precisely, if there is an $\varepsilon > 0$, such that for all x_0 satisfying $|x_0 - p| < \varepsilon$ we have $\lim_{n\to\infty} f^n(x_0) = p$. A fixed point p is called a source or a repelling fixed point if there is an $\varepsilon > 0$ such that for each point x_0 from the ε-neighborhood except for p itself there is an iterate x_k outside, i.e., $0 < |x_0 - p| < \varepsilon$ but $|x_k - p| > \varepsilon$ for some k. A period-k orbit of a point p is a periodic sink if p is a sink for the map f^k. Similarly, a period-k orbit of a point p is a periodic source if p is a source of f^k.

Theorem: Let $f: [0,1] \to [0,1]$ be continuously differentiable and p be a fixed point of f.

1. If $|f'(p)| < 1$, then p is a sink
2. If $|f'(p)| > 1$, then p is a source.

A graphic illustration of the theorem is as follows



Proof of Theorem: (Part 1 only). Let a be any number between $|f'(p)|$ and 1, $0 \le |f'(p)| < a < 1$. Since

$$\lim_{x \to p} \frac{|f(x) - f(p)|}{|x - p|} = |f'(p)| < a$$

there is a neighborhood $N_\varepsilon(p) = (p-\varepsilon, p+\varepsilon)$ for some $\varepsilon > 0$ so that

$$\frac{|f(x) - p|}{|x - p|} = \frac{|f(x) - f(p)|}{|x - p|} < a \quad (\text{Note: } f(p) = p)$$

for all x in $N_\varepsilon(p)$. In other words, $f(x)$ is closer to p than x is, by at least a factor of a. Thus, if $x \in N_\varepsilon(p)$, then $f(x) \in N_\varepsilon(p)$. By repeating the argument, we have $f^2(x), f^3(x), \cdots, f^n(x) \in N_\varepsilon(p)$ for all $n \ge 1$. In fact, we have

$$|f^n(x) - p| = |f(f^{n-1}(x)) - p| \le a|f^{n-1}(x) - p| \le \cdots \le a^n|x - p| \to 0$$

So p is a sink. □

Stability Test for Periodic Orbits: Let $\{p_1, p_2, \cdots, p_k\}$ denote a period-k orbit of f. Then by chain rule,

$$\begin{aligned}(f^k)'(p_1) &= (f(f^{k-1}))'(p_1) = f'(f^{k-1}(p_1))(f^{k-1})'(p_1) \\ &= f'(p_k) f'(f^{k-2}(p_1)) \cdots f'(p_1) \\ &= f'(p_k) f'(p_{k-1}) \cdots f'(p_1).\end{aligned}$$

Hence, $\{p_1, \cdots, p_k\}$ is a sink if

$$|f'(p_k) \cdots f'(p_1)| < 1$$

and a source if

$$|f'(p_k) \cdots f'(p_1)| > 1.$$

Example: Consider the logistic map

$$f_\lambda(x) = 4\lambda x(1-x), \quad 0 \le \lambda \le 1, \quad x \in [0, 1].$$

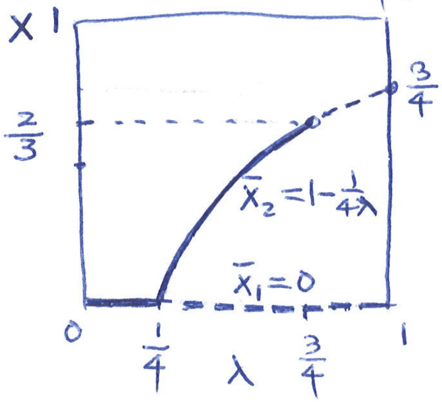
Let's find all the fixed points and period-2 points and determine their stability.

For fixed points, we solve $f_\lambda(x) = x$, or

$$4\lambda x(1-x) = x \Rightarrow \bar{x}_1 = 0, \text{ and } \bar{x}_2 = 1 - \frac{1}{4\lambda} \text{ for } \tfrac{1}{4} \le \lambda \le 1.$$

For their stability, consider $f'(x) = 4\lambda(1-2x)$ at $\bar{x}_{1,2}$ and we have $f'(\bar{x}_1) = 4\lambda$ and $f'(\bar{x}_2) = 4\lambda(1-2(1-\frac{1}{4\lambda})) = 2-4\lambda$. Thus

	sink	source	undetermined (change of stability)
$\bar{x}_1 = 0$	$0 \le \lambda < \frac{1}{4}$	$\frac{1}{4} < \lambda \le 1$	$f'(\bar{x}_1) = 1$, $\lambda = \frac{1}{4}$
$\bar{x}_2 = 1 - \frac{1}{4\lambda}$	$\frac{1}{4} < \lambda < \frac{3}{4}$	$\frac{3}{4} < \lambda \le 1$	$f'(\bar{x}_2) = 1$ @ $\lambda = \frac{1}{4}$ $f'(\bar{x}_2) = -1$ @ $\lambda = \frac{3}{4}$



The diagram ~~summ~~ summerize the result. The solid curves are for the stable (attracting) fixed points and the dashed curves are for the repelling fixed points

For period-2 points, we solve $f_\lambda^2(x) = f_\lambda(f_\lambda(x)) = x$ for those x which are not $\bar{x}_1, \bar{x}_2$. Note that both $\bar{x}_i$ are solutions to the fixed point problem $f_\lambda^2(x) = x$ for period-2 points. Let $g(x) = f_\lambda^2(x) - x = f_\lambda(f_\lambda(x)) - x = 4\lambda f_\lambda(x)(1 - f_\lambda(x)) - x$

$= 4\lambda(4\lambda x(1-x))(1 - 4\lambda x(1-x)) - x$. Factoring out the known factor $(x - \bar{x}_1)(x - \bar{x}_2)$, we have

$$\frac{g(x)}{(x-\bar{x}_1)(x-\bar{x}_2)} = 4\lambda\left[-16\lambda^2 x^2 + (16\lambda^2 + 4\lambda)x - (4\lambda + 1)\right] = 0$$

(You can use Maple to do the factorization). Solve ~~for~~ the quadratic equation, we have two solutions

$$\bar{y}_{1,2} = \frac{1}{8\lambda} + \frac{1}{2} \pm \frac{1}{8\lambda}\sqrt{-3 - 8\lambda + 16\lambda^2}.$$

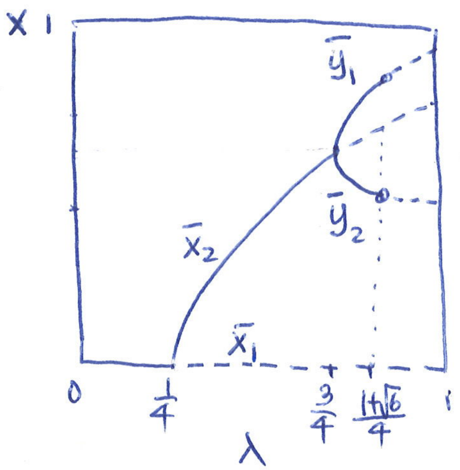
$\bar{y}_i$ are real if and only if $-3 - 8\lambda + 16\lambda^2 \ge 0$ which occurs in $[0,1]$ when $\frac{3}{4} \le \lambda \le 1$. Note at $\lambda = \frac{3}{4}$, $\bar{y}_1 = \bar{y}_2 = \frac{2}{3} = \bar{x}_2(\frac{3}{4})$.

For their stability, consider the stability test $f'(\bar{y}_1) f'(\bar{y}_2)$, and simplify it (again you can use maple to do this). This gives

$$f'(\bar{y}_1)f'(\bar{y}_2) = 4\lambda(1-2\bar{y}_1)\cdot 4\lambda(1-2\bar{y}_2) = -16\lambda^2+8\lambda+4, \ \tfrac{3}{4}\le\lambda\le 1.$$

Solving $-1 < f'(\bar{y}_1)f'(\bar{y}_2) = -16\lambda^2+8\lambda+4 < 1$ in $\lambda \in (\frac{3}{4}, 1)$ for attracting period-2 points, we find,

$$\frac{3}{4} < \lambda < \frac{1+\sqrt{6}}{4} \approx \frac{3.4495}{4}.$$



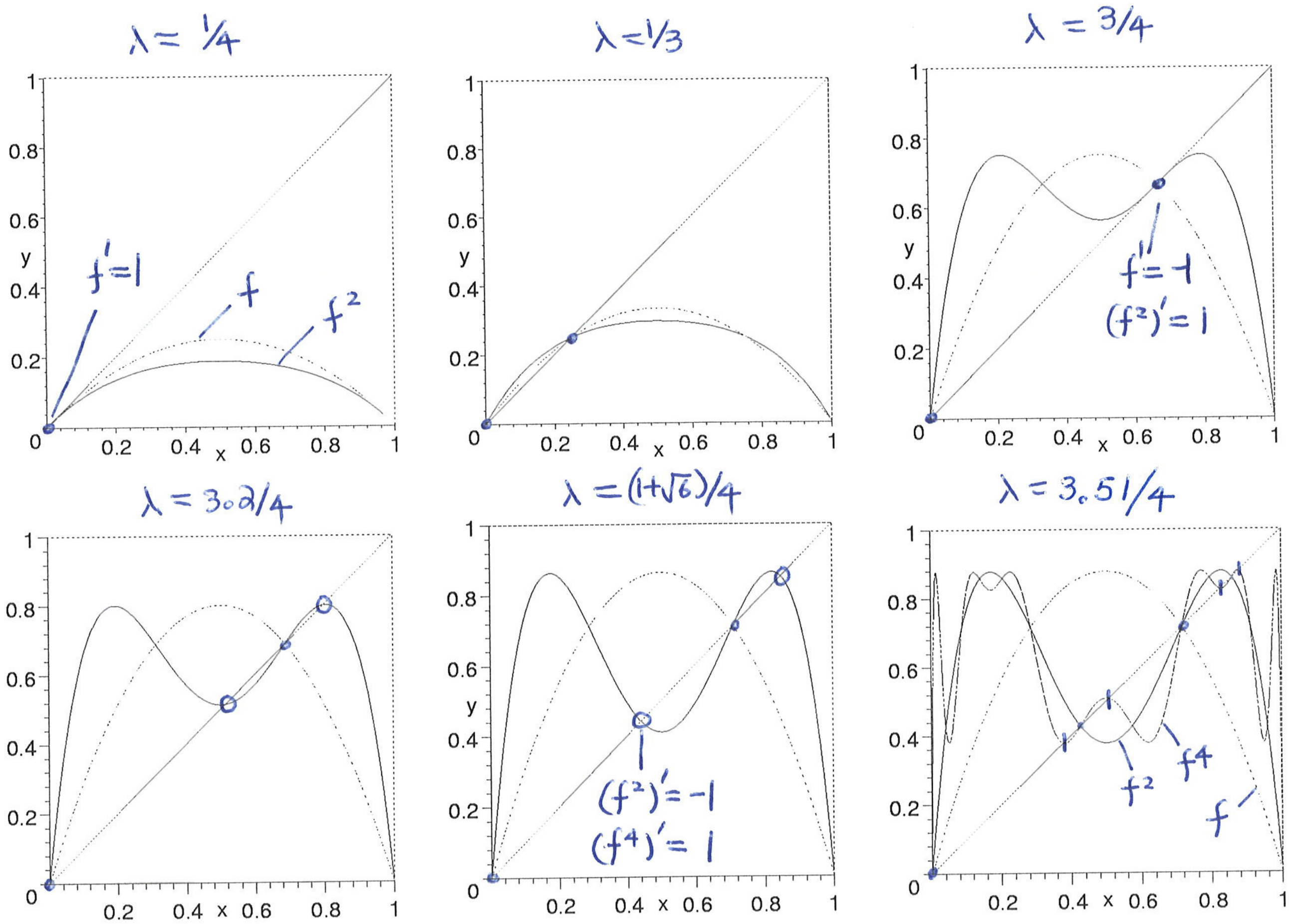
Updating the bifurcation diagram, we obtain the diagram on the left. Note that

$$\frac{d}{d\lambda}\bar{y}_1(\lambda)\Big|_{\lambda=\frac{3}{4}} = +\infty \text{ and } \frac{d}{d\lambda}\bar{y}_2(\lambda)\Big|_{\lambda=\frac{3}{4}} = -\infty$$

because of the fact that $-3-8\lambda+16\lambda^2=0$ at $\lambda=\frac{3}{4}$, which implies $\frac{d}{d\lambda}\sqrt{-3-8\lambda+16\lambda^2}\Big|_{\lambda=\frac{3}{4}} = +\infty$.

It is important to note a fact that the fixed point $\bar{x}_2$ looses its stability at the same point $\lambda=\frac{3}{4}$ at which two stable period-2 points are born. Also the fact that $f'(\bar{x}_2) = -1$ at $\bar{x}_2 = \frac{2}{3}$, $\lambda = \frac{3}{4}$. This phenomenon is referred to as period-doubling bifurcation. It is a typical behavior of fixed points having -1 at their derivatives. It is possible to show that at $\lambda = \frac{1+\sqrt{6}}{4}$, the condition $(f^2)'(\bar{y}_i) = -1$ holds and the lost of ~~the~~ stability for the period-2 points $\bar{y}_i$ at $\lambda = \frac{1+\sqrt{6}}{4}$ gives rise to a stable branch of period-4 points for some interval $\frac{1+\sqrt{6}}{4} < \lambda < \lambda_3 < 1$ for some $\lambda_3 < 1$.

This sequence of bifurcations is graphically illustrated in the next page.



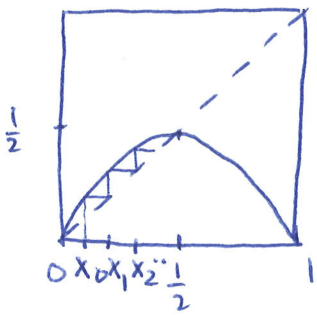
3. Basin of Attraction.

Let $f: I \to I$ be a map with $I = [0,1]$, or $\mathbb{R}$, or $[a,b]$, etc.

Let p be an attracting fixed point or attracting period-k point.

Definition: The basin of attraction of p, denoted by $B(p)$ is the largest interval containing p such that for every $x_0 \in B(p)$, $f^n(x_0) \to p$ if p is a fixed point or $f^n(x_0) \to \gamma(p) = \{p, f(p), \dots, f^{k-1}(p)\}$ if p is a period-k point.

Ex: Consider $f_\lambda(x) = 4\lambda x(1-x)$ with $\lambda = \frac{1}{2}$. Show that the basin of attraction for $p = 1/2$ is $B(\frac{1}{2}) = (0,1)$.



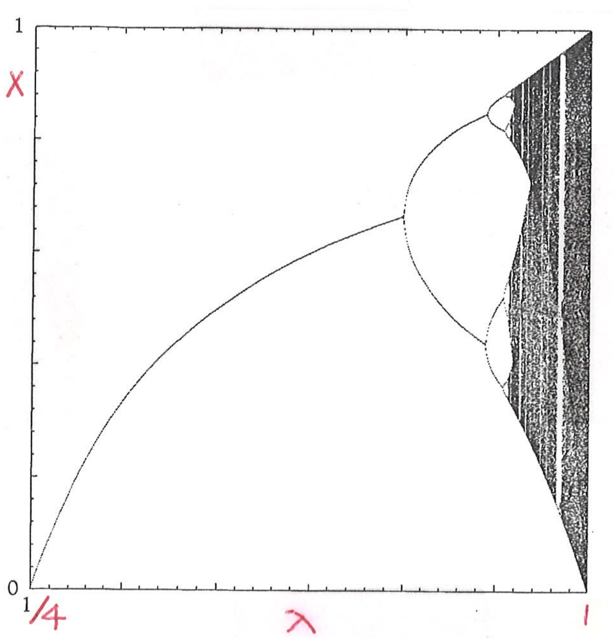
<u>Solution</u>: It is straightforward to check that for $f(x)=2x(1-x)$

(a) $f(x) < \frac{1}{2}$ for all $x \in [0,1]$, $x \neq \frac{1}{2}$;

(b) If $x \in (0, \frac{1}{2})$, then $x < f(x) < \frac{1}{2}$;

(c) If $x \in (\frac{1}{2}, 1)$, then $f(x) \in (0, \frac{1}{2})$.

So for every $x_0 \in (0, \frac{1}{2})$, the orbit $\gamma(x_0)$ defines an monotone increasing sequence $x_0 < x_1 < x_2 < \cdots < x_n < \cdots \frac{1}{2}$. Therefore $\lim_{n\to\infty} x_n$ exists and let $\bar{x}$ be the limit $\lim_{n\to\infty} x_n = \bar{x} \leq \frac{1}{2}$. Then $\bar{x}$ must be a fixed point in $(0, \frac{1}{2})$ (why?). Since $\frac{1}{2}$ is only fixed point in $(0, \frac{1}{2}]$, we conclude $\bar{x} = \frac{1}{2}$. If $x_0 \in (\frac{1}{2}, 1)$, then $x_1 \in (0, \frac{1}{2})$, The same argument applies, i.e. $x_n \to \frac{1}{2}$. Thus we have demonstrated that for every $x \in (0,1)$, $f^n(x) \to \frac{1}{2}$. It is obvious to see that $f(0)=0$, $f(1)=0$. That is the basin of attraction of $\frac{1}{2}$ is $B(\frac{1}{2}) = (0,1)$. $\square$



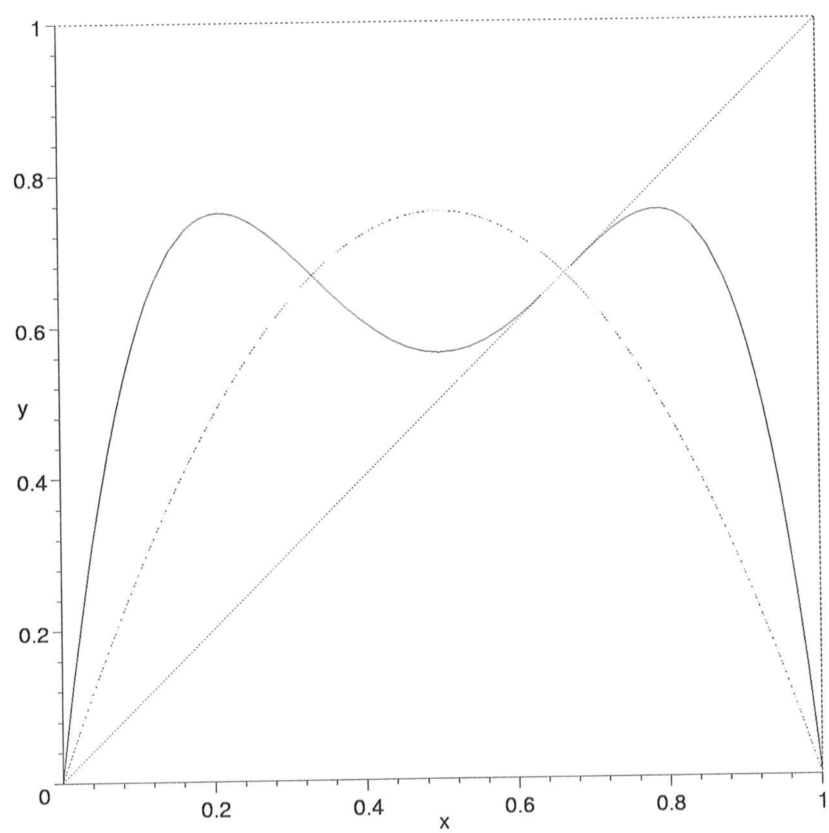
We can carry out similarly analysis on the question of basin of attraction for other attracting periodic points as well, though the task is not necessarily as easy. One effective tool is by numerical simulation to generate the so-called bifurcation diagram on the left. It was generated in steps as follows:

① Pick the number m of points for the parameter λ in say $[\frac{1}{4}, 1]$, and generate the point λ_i, $i=1,2,\cdots,m$. ② For each λ_i, randomly pick an $x_0 \in (0,1]$ and generate the 1st N iterates. ③ Plot only the last $n \ll N$ iterates.

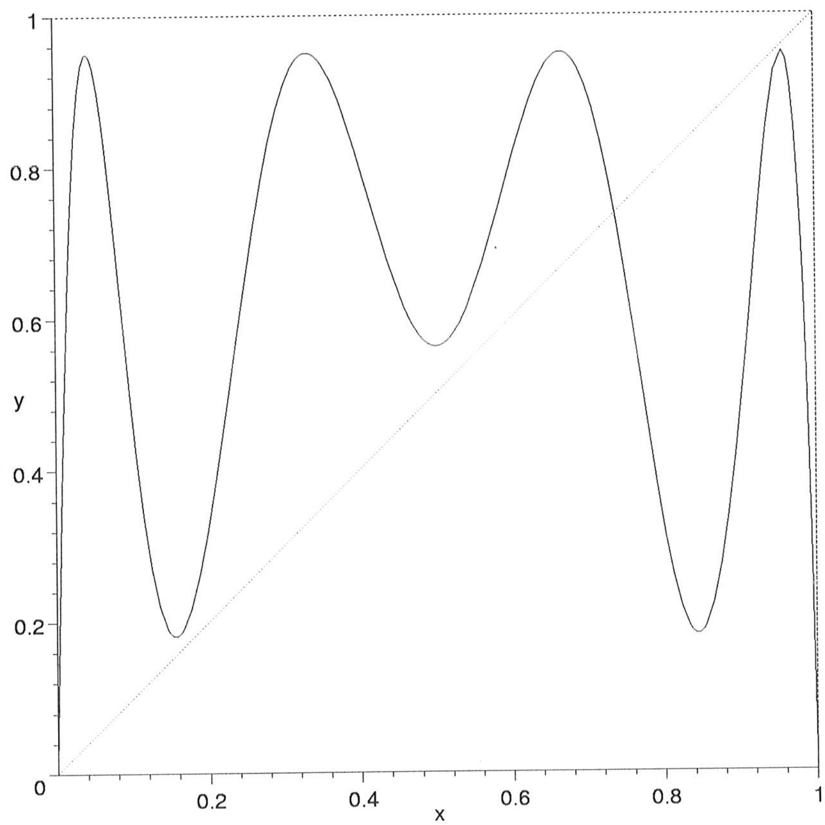
```
> #
> # Iterate Plot for Logistic Family.
> #
> with(plots):
> f:=(lmd,x)->4*lmd*x*(1-x); #Define the Logistic family.
```

$$f := (lmd, x) \to 4\, lmd\, x\, (1-x)$$

```
> p1:=plot({[1,0],[1,1]},x=0..1,y=0..1):p2:=plot(x,x=0..1,y=0..1):p3
  :=plot(1,x=0..1,y=0..1): # Plot the box 0<=x<=1,0<=y<=1 and the
  diagonal y=x.
> lmd:=3/4:iterate1:=plot(f(lmd,x),x=0..1,y=0..1,color=green):iterat
  e2:=plot(f(lmd,f(lmd,x)),x=0..1,y=0..1,color=black):display({p1,p2
  ,p3,iterate1,iterate2}); # Plot the first and the second iterates
  of f for a given 'lmd'.
```



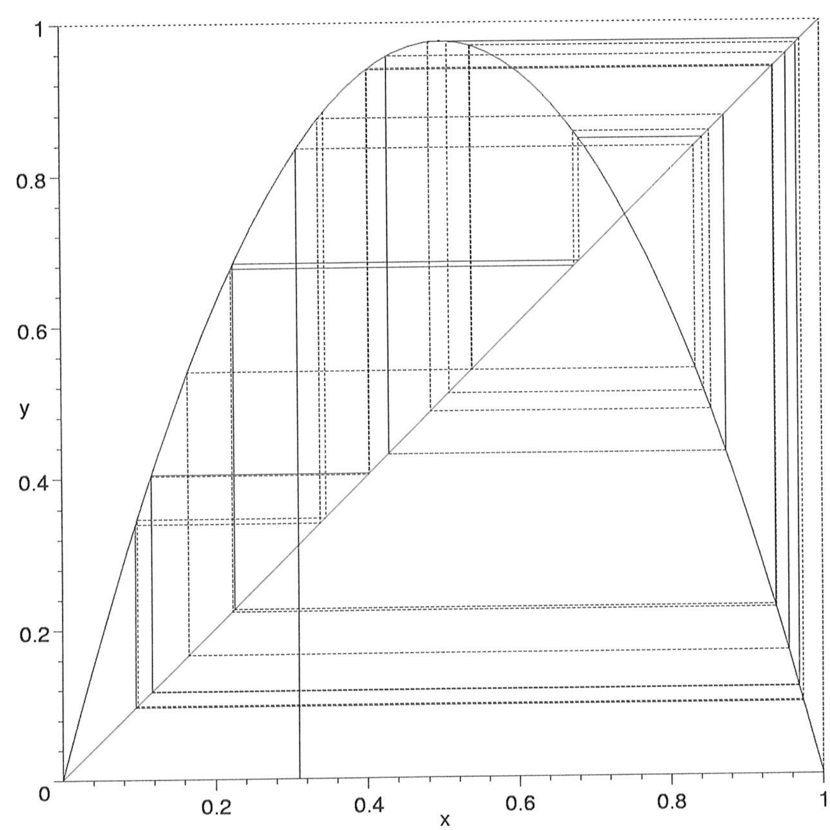

```
> lmd:=3.8/4:iterate3:=plot(f(lmd,f(lmd,f(lmd,x))),x=0..1,y=0..1,col
  or=black):display({p1,p2,p3,iterate3}); # No period-3 points yet
  for 'lmd'<3.8/4.
```



1
0.8
0.6
y
0.4
0.2
0
0.2
0.4
0.6
0.8
1
x

[>

```
> #
> # Cobweb Plot for Logistic Family.
> #
> with(plots):
> lmd:=3.9/4:N:=30: # Choose the parameter value and the number of
  iterates.
> f:=(lmd,x)->4*lmd*x*(1-x): #Define the Logistic family.
> p1:=plot({[1,0],[1,1]},x=0..1,y=0..1):p2:=plot(x,x=0..1,y=0..1):p3
  :=plot(1,x=0..1,y=0..1): # Plot the box 0<=x<=1,0<=y<=1 and the
  diagonal y=x.
> iterate1:=plot(f(lmd,x),x=0..1,y=0..1,color=black): # Plot the
  graph f.
> x0:=evalf(rand()/10^12):x1:=f(lmd,x0):P:=[[x0,0],[x0,x1]]:
  #Randomly initialize a point and its 1st iterate.
> for n from 1 to N-1 do
  x0:=x1:x1:=f(lmd,x0):P:=[op(P),[x0,x0],[x0,x1]]:od: #Generate the
  points of cobweb plot.
> cobweb:=plot(P,color=blue):
> display({p1,p2,p3,iterate1,cobweb}); # The Cobweb plot.
```

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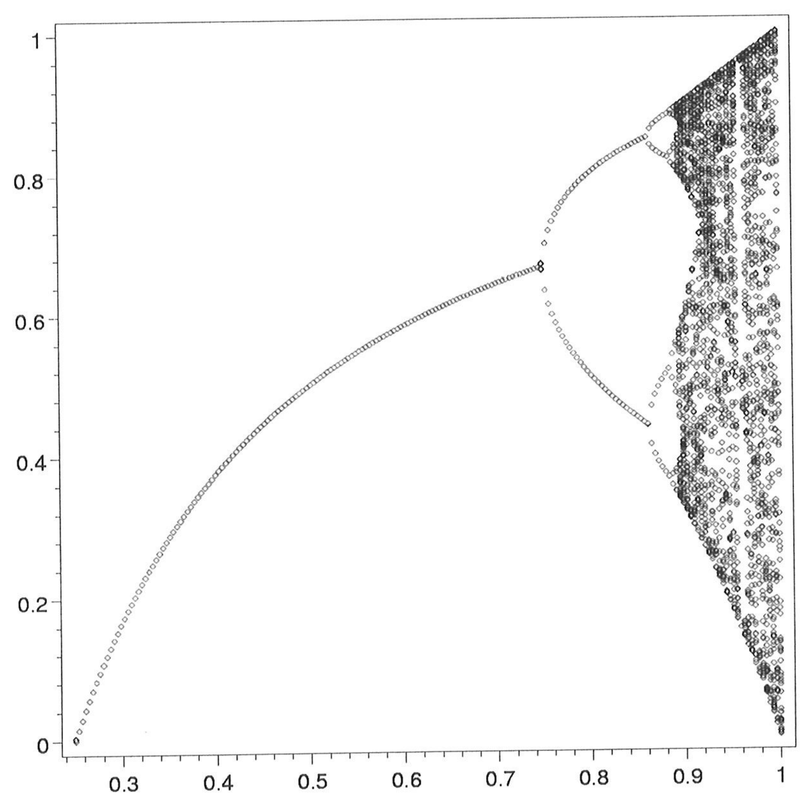
> #
> # Bifurcation Diagram for Logistic Family.
> #
> with(plots):
> f:=(lmd,x)->4*lmd*x*(1-x):
> a:=1/4:b:=1:m:=200:hn:=300:tn:=100:# To run, choose the parameter
  range '[a,b]' and the number 'm' for parameter points, the number
  'hn' for transcient itereates, and the number 'tn' for the
  actually plotted iterates. I.e., the trunkated number of the
  numerical orbit is 'hn+tn', but only the tail end 'tn' points are
  plotted to capture the underlining attractor structure.
> P:=[[a,0]]:for i from 1 to m+1 do lmd:=a+(i-1)*(b-a)/m:
  x0:=evalf(rand()/10^12):
  for j from 1 to hn do x0:=f(lmd,x0): od: for k from 1 to tn do
  P:=[op(P),[lmd,x0]]:x0:=f(lmd,x0):od:od:
> pointplot(P,symbol=point,axes=BOXED);
```

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>
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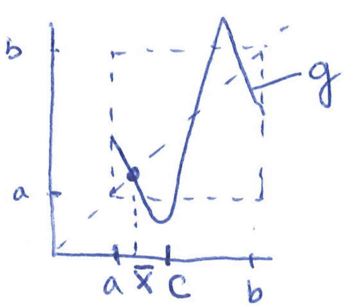
4. Period-3 Points and Sharkovskii's Ordering

In early 70s, T. Y. Li and J. York discovered a curious result of the following

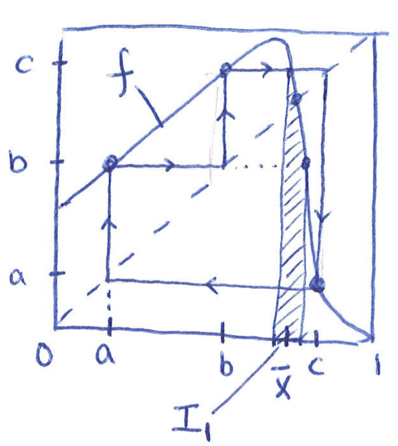
Theorem: Let $f: [0,1] \to [0,1]$ be a continuous map and f has a period-3 point. Then f has periodic points of all period.

they published the result in a paper title "Period Three Implies Chaos". The field of chaos derives its name from this paper. To prove this result we first prove an elementary result from calculus based on intermediate value theorem.

Lemma: Let g be a continuous map whose image of an interval $[a,b]$ covers $[a,b]$, i.e. $g([a,b]) \supset [a,b]$. Then g must have a fixed point.



Proof: If either $g(a)=a$ or $g(b)=b$, we are done. Otherwise, assume $g(a)\neq a$, $g(b)\neq b$. For definiteness we assume $g(a)>a$. (The other case is handled exactly the same way.) Because $g([a,b])$ covers $[a,b]$; there must be a point $a<c\leq b$ so that $g(c)\leq a<c$. To consider the fixed point problem $g(x)=x$, it is equivalent to solving the equation $h(x):=g(x)-x=0$. Because g is continuous, so is h. In addition, $h(a)=g(a)-a>0$ and $h(c)=g(c)-c<0$ by the choice of c. So by the intermediate value theorem, there must be a point $\bar{x}\in(a,c)\subset[a,b]$ such that $h(\bar{x})=0$, i.e. $g(\bar{x})=\bar{x}$. This proves the theorem. $\square$.



Proof of the Theorem: First we assume $0 \le a < b < c \le 1$ are the period-3 points, i.e. $f(a) = b$, $f(b) = c$, $f(c) = a$. The other configurations of a, b, c can be proved similarly. The proof is constructed by making use of Lemma. Now, we proceed to prove the existence of period-1 (fixed point), period-2, and period-k for any $k \ge 4$. For fixed point, consider the interval $[b, c]$. Since $f(b) = c$, $f(c) = a < b$, $f([b,c]) \supseteq [a, c] \supset [b, c]$. That is $f([b, c])$ covers $[b, c]$. By Lemma, there is a fixed point $\bar{x}$. The first case is done. For the second case, consider $g(x) = f^2(x)$ on interval $[a, b]$. The following mapping ~~d~~ diagram shows $g([a, b])$ covers $[a, b]$:

$$[a, b] \overset{f}{\supseteqq} [b, c] \overset{f}{\supseteqq} [a, c].$$

The fixed point $\bar{y}$ of g is not a nor b, that is $a < \bar{y} < b$. And $f(\bar{y}) \in [b, c]$. So $\bar{y}$ is not a fixed point of f. Therefore $\bar{y}$ must be a period-2 point of f. This proves the second case. Last, consider the general case of period-k points with $k \ge 4$. Note that because $f([b, c]) \supseteq [a, c] \supset [b, c]$, there must be an interval $I_1 \subset [b, c]$ so that $f(I_1) = [b, c]$, i.e. I_1 is a preimage of $[b, c]$ from $[b, c]$. Notice the same relation $f(I_1) \supset I_1$ and thus the same argument applies to get $I_2 \subseteq I_1$, $f(I_2) = I_1$, i.e. $f^2(I_2) = [b, c]$. By induction, we can have $I_{k-2} \subseteq I_{k-3} \subseteq \cdots \subseteq I_2 \subset I_1$ so that $f^{k-2}(I_{k-2}) = [b, c]$. Therefore $f^{k-1}(I_{k-2}) = f(f^{k-2}(I_{k-2})) = f([b, c]) \supseteq [a, c] \supset [a, b]$. So there must be a preimage interval $K \subset I_{k-2} \subset [b, c]$ such that $f^{k-1}(K) = [a, b]$. Last $f^k(K) = f(f^{k-1}(K)) = f([a, b]) \supseteq [b, c] \supset K$. By Lemma, there is a $\bar{z} \in K$ s.t. $f^k(\bar{z}) = \bar{z}$. $\bar{z}$ is a desired period-k point. □

Mathematics is often rediscovered and the second time does it make an impact. This is the case with the period-3-implies-chaos result. In the early '60's, a Rassian mathematician A.N. Sharkovskii's obtained a much general result which few notices at that time. The result is stated as follows.

<u>Theorem</u>: Let $f: [0,1] \to [0,1]$ be continuous. Let's define an order of the natural numbers as follows:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd \cdots \rhd 2^n \cdot 3 \rhd 2^n \cdot 5 \rhd 2^n \cdot 7 \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd \cdots \rhd \cdots \rhd 8 \rhd 4 \rhd 2 \rhd 1$$

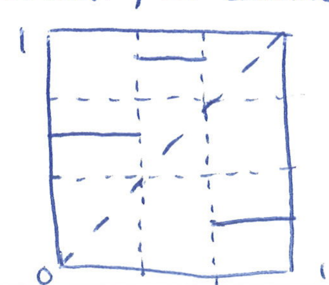
Then if f has a period-k point, f must have periodic points of all periods $l \lhd k$ in ordering above. □

Sharkovskii's result is the best that can be in the sense that one can construct continuous maps which has all period-l point for $l \lhd k$ but no period-m points for $m \rhd k$. Li and York's result is only a special case of Sharkovskii's.

<u>HWK</u> (Due Next Friday, 14 April). 1) Find a period-3 point a of the tent map $T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \le 1 \end{cases}$ satisfying: $a < T(a) < \frac{1}{2} < T^2(a)$.

2) Show that ~~all~~ period-k points of all $k \geq 1$ are repellers for the tent map T.

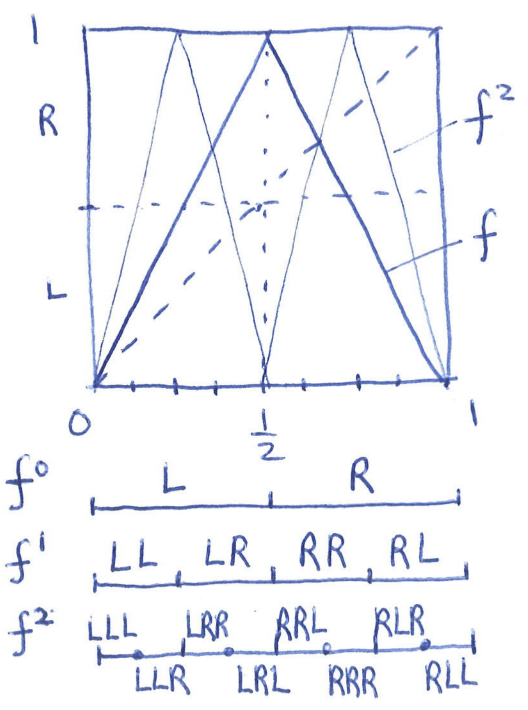
3) The Continuity condition of f is sufficient, It cannot be omitted. For example, the map of the diagram has only period-3 point and no other period-k point.



5. Itineraries

One important aspect of 1-dimensional dynamics is about the structure of orbits. One can ask or be interested in the kind of orbits a particular map may have. In addition to the concept of fixed point, periodic point, there are other kinds of orbits as well. For example, an orbit $\gamma(x_0)$ of a map f is eventually periodic if there is a $k>0$ such that $f^k(x_0)$ is a periodic orbit. A map may even have orbits that are not periodic, nor eventually periodic. The concept of chaotic orbit, to be defined later, is one of such examples. In this section, we will discuss a useful concept to keep track of orbits. We will illustrate the idea with the tent map

$$f(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1. \end{cases}$$



The tent map has a unique critical point $x=\frac{1}{2}$. It devides the interval $I=[0,1]$ into two subintervals: $L=[0,\frac{1}{2})$ the left interval and $R=[\frac{1}{2},1]$, the right interval. (Assigning the critical point $x=\frac{1}{2}$ to either R or L is not essential.) On L, f is increasing while on R, f is decreasing.

For any orbit $\gamma(x_0) = \{x_0, x_1, x_2, \cdots\}$, there is a unique symbol sequence $S(x)$ in Ls and Rs associated with $\gamma(x_0)$ that is called the itinerary of x_0. $S(x_0) = S_0 S_1 S_2 \cdots$, with

$$S_i = \begin{cases} L, & \text{if } x_i = f^i(x_0) \in L \\ R, & \text{if } x_i = f^i(x_0) \in R. \end{cases}$$

For example, the itinerary of the fixed point $x=0$ is $S(0) = LLL\cdots := \bar{L}$, a repeating sequence in L. That of the critical point $x=\frac{1}{2}$ is $S(\frac{1}{2}) = RRLLL\cdots = RR\bar{L} := RK_f$, where $K_f = R\bar{L}$ is called the kneading sequence of f. Note that $K_f = S(f(\frac{1}{2}))$, the itinerary of the maximum value of f. Note that, had we defined $L=[0,\frac{1}{2}]$, $R=(\frac{1}{2},1]$, then $S(\frac{1}{2})$ would be $S(\frac{1}{2}) = LRLL\cdots = LK_f$. In other words, with the definition $L=[0,\frac{1}{2})$, $R=[\frac{1}{2},1]$, any sequence S that ends with LK_f is not an itinerary of any orbit. The question is, is a sequence S that does not end with LK_f the itinerary of an orbit? The answer is Yes as stated in the Proposition below.

<u>Proposition</u>: Let $L=[0,\frac{1}{2})$, $R=[\frac{1}{2},1]$. A sequence $S=S_0S_1\cdots$ in Ls and Rs is an itinerary for the tent map if and only if S does not end with the sequence LK_f.

This result gives us a simple, symbolic way to map out all orbits. For examples, a periodic orbit has a repeating symbol segment for its itinerary, $S = \overline{s_0 s_1 \cdots s_{k-1}}$; an eventually periodic orbit ~~has a~~ ends with a repeating segment after a transient, finite symbol sequence, $S = s_0 s_1 \cdots s_n \overline{s_{n+1} \cdots s_{n+k}}$. In this way, any point x from $[0, 1]$ can be uniquely identified with its itinerary $S(x)$, which is necessarily an infinite sequence.

The device of itinerary can also be used to identify a set of points. For example, we would like to ask: what set of points whose itinerary begins with, say, LR? These points share the property of beginning in the L and being mapped to the R. As illustrated in the diagram, a finite itinerary sequence is uniquely identified with an interval. For example, $LR = [\frac{1}{4}, \frac{1}{2})$, $LLL = [0, \frac{1}{8}]$, and so on.

Proof of Proposition: Let $S = s_0 s_1 \cdots s_n \cdots$ with $s_i = L$ or R, not ending with LK_f. The question is, is there a unique point $\bar{x} \in I$ such that $\bar{x} \in S_0$, $f(\bar{x}) \in S_1$, $f^2(\bar{x}) \in S_2$, ... and so on? Formulating differently, let

$$I_{s_0 s_1 \cdots s_n} = \{x : f^i(x) \in S_i,\ i = 0, 1, 2, \ldots, n\},$$

i.e. the set of points whose 1st n iterates are in $S_1, S_2, \cdots, S_n$ and starting in S_0. Then we automatically have

$$I_{s_0} \supset I_{s_0 s_1} \supset \cdots \supset I_{s_0 s_1 \cdots s_n} \supset I_{s_0 s_1 \cdots s_{n+1}} \supset \cdots,$$

a nested sequence of sets and our sought-after point $\bar{x}$ shall be in all the sequential sets:

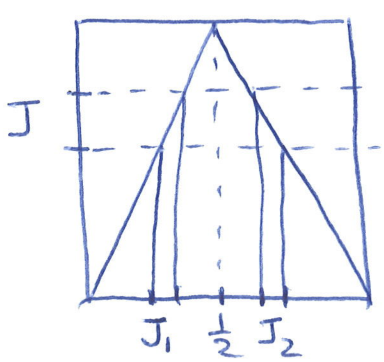
$$\bar{x} \in \bigcap_{n=0}^{\infty} I_{s_0 s_1 \cdots s_n}$$

In other words, we have to show such nested sets are each non-empty and the intersection is a single point! We now use induction to facilitate the argument. We claim

1) $I_{s_0 s_1 \cdots s_n}$ is a non-empty interval with length
$$|I_{s_0 s_1 \cdots s_n}| = \left(\tfrac{1}{2}\right)^{n+1}$$

2) $f(I_{s_0 s_1 \cdots s_n}) = I_{s_1 s_2 \cdots s_n}$, i.e.
$$I_{s_0 s_1 \cdots s_n} = I_{s_0} \cap f^{-1}(I_{s_1 \cdots s_n}).$$

The case with $n=0$ is trivial as $I_{s_0} = L$ or R, $I_\phi = [0,1]$, where $s_1 \cdots s_0$ is an empty sequence ϕ. Assume the case for n. Now by induction we only need to prove the $n+1$ case. By induction, $I_{s_1 s_2 \cdots s_{n+1}}$ is an interval of length $|I_{s_1 s_2 \cdots s_{n+1}}| = \left(\frac{1}{2}\right)^{n+1}$. Because for every interval J of $[0,1]$, there are precisely two pre-image intervals under f, one is in L and the other is in R. Because f expands each interval by a factor of 2, the preimage is contracted by the inverse of the factor. Applying this argument to $J = I_{s_1 \cdots s_{n+1}}$



we obtain $I_{s_0} \cap f^{-1}(I_{s_1 \cdots s_{n+1}})$ is an interval of length $(\frac{1}{2})^{n+2}$ and

$$I_{s_0} \cap f^{-1}(I_{s_1 \cdots s_{n+1}}) = \{x : x \in I_{s_0},\ f(x) \in I_{s_1 \cdots s_{n+1}}\}$$
$$= \{x : x \in I_{s_0},\ f^i(x) \in S_i,\ i = 1, 2, \cdots, n+1\}$$
$$= I_{s_0 s_1 \cdots s_{n+1}}, \text{ by definition.}$$

This proves the claim.

To complete the proof, we take the closure of ($\overline{I_{s_0 \cdots s_n}}$) $I_{s_0 s_1 \cdots s_n}$ and then intersect. By a result from calculus, $\bigcap \overline{I_{s_0 \cdots s_n}} = \{\bar{x}\}$ contains a unique point. The last thing to show is $S(\bar{x}) = s_0 s_1 \cdots s_n \cdots$, the itinerary of $\bar{x}$ is $s_0 s_1 \cdots s_n \cdots$. Since by taking the closure $\overline{I_{s_0 \cdots s_n}}$ we actually allow sequences ending with LK_f which count each eventually fixed point $\bar{I}$ exactly twice. Thus, by excluding sequences ending with LK_f, the correspondence between the points and their itineraries is uniquely defined. □

6. Sensitive Dependence on Initial Conditions

Definition: Let $f: I \to I$ be a map. A point x_0 has sensitive dependence on initial conditions if there exists a nonzero distance $d > 0$ such that for every small neighborhood $N_\varepsilon = (x_0 - \varepsilon, x_0 + \varepsilon)$ for however small $\varepsilon > 0$ it contains a point $x \in N_\varepsilon$ so that $|f^k(x) - f^k(x_0)| \geq d$ for some nonnegative integer k.

Example: Every point x_0 of $[0,1]$ has sensitive dependence on initial conditions under the tent map

$$f(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

This fact can be easily verified by using the itinerary. More specificly, let $S(x_0) = s_0 s_1 s_2 \cdots$ be its itinerary. We will demenstrate the case with the separation distance $d = \frac{1}{4}$. For any $\varepsilon > 0$, let n be an integer such $(\frac{1}{2})^{n+1} < \varepsilon$. Because interval $I_{s_0 s_1 \cdots s_n} \ni x_0$ and $|I_{s_0 s_1 \cdots s_n}| = (\frac{1}{2})^{n+1}$, we must have $I_{s_0 s_1 \cdots s_n} \subset N_\varepsilon = (x_0 - \varepsilon, x_0 + \varepsilon)$. Since $\cancel{f^{n+1}(x_0)}$ $S(x_0) = s_0 s_1 \cdots s_n s_{n+1} s_{n+2} \cdots$, $S(f^{n+1}(x_0)) = s_{n+1} s_{n+2} \cdots$ which implies $f^{n+1}(x_0) \in I_{s_{n+1} s_{n+2}}$. There are 4 intervals (of length $\frac{1}{4}$ each) $I_{LL}, I_{LR}, I_{RR}, I_{RL}$ and one of them is $I_{s_{n+1} s_{n+2}}$. We now only need to choose symbols $s'_{n+1} s'_{n+2}$ so that $I_{s_{n+1} s_{n+2}}$ and $I_{s'_{n+1} s'_{n+2}}$ are sperated by one of these 4 intervals. Then we choose a point x' whose itineraries differ from x_0's only at the $n+1$ and $n+2$ slots, i.e. $S(x') = s_0 s_1 \cdots s_n s'_{n+1} s'_{n+2} s_{n+3} \cdots$. Then both $x_0, x' \in I_{s_0 \cdots s_n} \subset N_\varepsilon$ but $f^{n+1}(x_0) \in I_{s_{n+1} s_{n+2}}$, $f^{n+1}(x') \in I_{s'_{n+1} s'_{n+2}}$, implying $|f^{n+1}(x_0) - f^{n+1}(x')| \geq \frac{1}{4} = d$.

7. Lyapunov Exponents

Assume $f: I \to I$ is differentiable everywhere except for a few point. A way to measure the sensitive dependence on initial conditions is ~~to~~ to measure the rate of separation along an orbit. Assume x_0 & y_0 are very close, by chain rule and the mean value theorem of calculus, we have

$$\begin{aligned} |f^n(y_0) - f^n(x_0)| &= |f(f^{n-1}(y_0)) - f(f^{n-1}(y_0))| \\ &\approx |f'(f^{n-1}(x_0))|\,|f^{n-1}(y_0) - f^{n-1}(x_0)| \\ &\approx \cdots \approx |f'(x_{n-1})||f'(x_{n-2})|\cdots|f'(x_1)||f'(x_0)|\,|y_0 - x_0|. \end{aligned}$$

That is the product $\prod_{i=0}^{n-1}|f'(x_i)|$ roughly measures the accumulative separation effect of the orbit $\gamma(x_0)$. In general, we expect this accumulative separation is compounded on average by a single factor $L > 1$, i.e.

$$\prod_{i=0}^{n-1}|f'(x_i)| \approx L^n \quad \text{or} \quad L \approx \left(\prod_{i=0}^{n-1}|f'(x_i)|\right)^{\frac{1}{n}}.$$

This leads to our next definition.

Definition: If f is differentiable at every iterate of an ~~the~~ orbit $\gamma(x_0) = \{x_0, x_1, x_2, \cdots\}$. The Lyapunov number $L(x_0)$ is defined as

$$L(x_0) = \lim_{n\to\infty}\left(\prod_{i=0}^{n-1}|f'(x_i)|\right)^{\frac{1}{n}},$$

if the limit exists. The Lyapunov exponent $h(x_0)$ is defined as

$$h(x_0) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|$$

if the limit exists. Notice that h exists iff L exists, and $\ln L = h$. Also, $h > 0$ iff $L > 1$.

Example: If x_0 is not eventually the fixed point $x=0$ of the tent map $f(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$, i.e., the itinerary $S(x_0)$ does not end with K_f, the reading sequence, then f is differentiable along the orbit $\gamma(x_0)$ and $|f'(x_i)| = 2$ for every x_i. Thus the Lyapunov number is

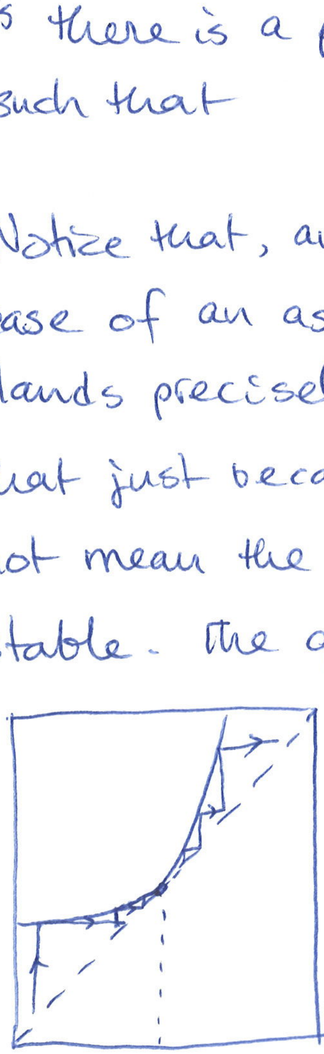
$$L(x_0) = \lim_{n\to\infty} \left(\prod_{i=0}^{n-1} |f'(x_i)| \right)^{\frac{1}{n}} = \lim_{n\to\infty} 2 = 2$$

as expected. The Lyapunov exponent $h(x_0) = \ln 2$ as $L = e^h$.

Definition: An orbit $\gamma(x_0) = \{x_0, x_1, \ldots\}$ is called asymptotically periodic if it converges to a periodic orbit as $n \to \infty$. That is there is a periodic orbit $\{y_0, y_1, \ldots, y_{k-1}, y_0, y_1, \ldots, y_{k-1}, \ldots\}$ such that

$$\lim_{n\to\infty} |x_n - y_n| = 0. \quad \square$$

Notice that, an eventually periodic orbit is an extreme case of an asymptotically periodic orbit, where the orbit lands precisely on a periodic orbit. We should be aware that just because an orbit is asymptotically periodic does not mean the associated periodic orbit is asymptotically stable. The graph illustrates a unstable fixed point which is only asymptotically stable on one side.



HW: (Due next Friday, April 21). Write a program to calculate the Lyapunov exponent of $f_\lambda(x) = 4\lambda x(1-x)$ for values of the parameter λ between $\frac{1}{2}$ and 1. Graph the results as a function of λ.