Sub-Foliation of Left Manifold

The Sub-Foliation of Left-Manifold Theorem obtained in this note can be used to obtained all commonly encountered invariant foliations for both diffeomorphisms and ordinary differential equations. They include: the strong-stable foliation of the stable manifold, $\mathcal{F}^{ss}_{\text{loc}} \subset W^{\text{s}}_{\text{loc}}$, the strong-stable foliation of the center-stable manifold, $\mathcal{F}^{ss}_{\text{loc}} \subset W^{\text{cs}}_{\text{loc}}$, the stable foliation of the center-stable manifold, $\mathcal{F}^{s}_{\text{loc}} \subset W^{\text{cs}}_{\text{loc}}$, the unstable foliation of the center-unstable manifold, $\mathcal{F}^{u}_{\text{loc}} \subset W^{\text{cu}}_{\text{loc}}$, the strong-unstable foliation of the center-unstable manifold, $\mathcal{F}^{uu}_{\text{loc}} \subset W^{\text{cu}}_{\text{loc}}$, and the strong-unstable foliation of the unstable manifold, $\mathcal{F}^{uu}_{\text{loc}} \subset W^{\text{u}}_{\text{loc}}$. By using the stable foliation of the center-stable manifold and the unstable foliation of the center-unstable manifold, one can prove the uniqueness of center-manifold dynamics.

Let $\bar{q}$ be a fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $J = Df(\bar{q})$.

Definition 1. *Let $[\lambda_1, \lambda_2]$ and $[\mu_1, \mu_2]$ be two pseudo-hyperbolic splits for J. The μ-split is called a sub-tight split if*

(i) $[\mu_1, \mu_2] \le [\lambda_1, \lambda_2]$, *i.e.,* $\mu_2 \le \lambda_1$.

(ii) $\mu_1 \lambda_1 < \mu_2$.

The λ-split is called a sup-tight split if (i) holds and $\lambda_1 < \lambda_2 \mu_2$.

Denote by $\mathbb{E}^{\lambda_1}$ the generalized eigenspace of J for eigenvalues $\sigma^1 = \{\lambda \in \sigma : |\lambda| \le \lambda_1\}$ and $\mathbb{E}^{\lambda_2}$ the generalized eigenspace of J for eigenvalues $\sigma^2 = \{\lambda \in \sigma : |\lambda| \ge \lambda_2\}$. Then $\mathbb{R}^d \cong \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\lambda_2}$. Similarly, we have $\mathbb{R}^d \cong \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$, with $\mathbb{E}^{\mu_1} \subset \mathbb{E}^{\lambda_1}$ and $\mathbb{E}^{\lambda_2} \subset \mathbb{E}^{\mu_2}$. For point $p = (x, y) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ we will use $\pi_{\mu_1} : \mathbb{R}^d \to \mathbb{E}^{\mu_1}$ for the coordinate projection map with $\pi_{\mu_1}(p) = x$. Similarly, we have $\pi_{\mu_2} : \mathbb{R}^d \to \mathbb{E}^{\mu_2}$ with $\pi_{\mu_2}(p) = y$. Also, for the λ-splitting, we have for $q = (u, w) \in \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\lambda_2}$, $\pi_{\lambda_1} : \mathbb{R}^d \to \mathbb{E}^{\lambda_1}$ with $\pi_{\lambda_1}(q) = u$ and $\pi_{\lambda_2} : \mathbb{R}^d \to \mathbb{E}^{\lambda_2}$ with $\pi_{\lambda_2}(q) = w$. We will use exclusively $p = (x, y)$ for μ-splitted points and q for λ-splitted points. We will write $p = \pi_\lambda(p) = (\pi_{\lambda_1}(p), \pi_{\lambda_2}(p)) \in \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\lambda_2}$ if we want to resolve p in the λ-splitting, and $q = \pi_\mu(q) = (\pi_{\mu_1}(q), \pi_{\mu_2}(q))$ or $q = (x_q, y_q) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ if we want to resolve q in the μ-splitting. Also, we can use π_i for subspace of $\mathbb{R}^d$ that contains $\mathbb{E}^i$. For example, $\mathbb{E}^{\mu_1}$ is a subspace of $\mathbb{E}^{\lambda_1}$, so if we let $\mathbb{F}$ be any transversal complement of $\mathbb{E}^{\mu_1}$ in $\mathbb{E}^{\lambda_1}$, then for any $u \in \mathbb{E}^{\lambda_1} \cong \mathbb{E}^{\mu_1} \times \mathbb{F}$, $\pi_{\mu_1}(u) \in \mathbb{E}^{\mu_1}$ is perfectly defined. Similarly, since $\mathbb{E}^{\lambda_2} \subset \mathbb{E}^{\mu_2}$, for any $y \in \mathbb{E}^{\mu_2}$, $\pi_{\lambda_2}(y) \in \mathbb{E}^{\lambda_2}$ is well-defined by the same reason.

Definition 2. *Let $\bar{q}$ be a fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $[\lambda_1, \lambda_2]$ be a pseudo-hyperbolic splits for $J = Df(\bar{q})$ and $[\mu_1, \mu_2]$ be a sub-tight split of $[\lambda_1, \lambda_2]$. Let $\alpha,\ \beta$ be any constants satisfying*

$$\mu_1 < \alpha < \mu_2 \le \lambda_1 < \beta < \lambda_2.$$

Let $W^{\lambda_1} = \{p : \sup\{\beta^{-n}[f^n(p) - \bar{q}] : n \geq 0\} < \infty\}$ *be the lambda-left manifold of* $\bar{q}$*. For every* $q \in W^{\lambda_1}$ *the sub-fiber of* q *is defined as*

$$\mathcal{F}^\mu(q) = \{p \in W^{\lambda_1} : \sup\{\alpha^{-n}[f^n(p) - f^n(q)] : n \geq 0\} < \infty\}$$

and the collection

$$\mathcal{F}^\mu = \{\mathcal{F}^\mu(q) : q \in W^{\lambda_1}\}$$

is called the sub-foliation of W^{λ_1}*.*

Notice that the sub-fiber defines an equivalence relation on W^{λ_1}: $q \in \mathcal{F}^\mu(q)$; $p \in \mathcal{F}^\mu(q)$ iff $q \in \mathcal{F}^\mu(p)$ and $\mathcal{F}^\mu(q) = \mathcal{F}^\mu(p)$. Also, the foliation is an invariant family with

$$f(\mathcal{F}^\mu(q)) = \mathcal{F}^\mu(f(q)).$$

Also W^{λ_1} can be filled by fibers through an invariant sub-manifold of W^{λ_1} as a stem that runs transverse to $\mathcal{F}^\mu$. In addition, the mu-left manifold is the fiber through $\bar{q}$, $W^{\mu_1} = \mathcal{F}^\mu(\bar{q})$.

Theorem 1 (Sub-Foliation of Left-Manifold Theorem)**.** *Let* $\bar{q}$ *be a fixed point of a* $C^{1,1}$ *diffeomorphism* f*. Let* $[\lambda_1, \lambda_2]$ *be a pseudo-hyperbolic splits for* $J = Df(\bar{q})$ *and* $[\mu_1, \mu_2]$ *be a sub-tight split of* $[\lambda_1, \lambda_2]$*. Then a sufficiently small* $\|f - Df(\bar{q})\|_1$ *implies there is a* C^1 *function*

$$\psi_2 : \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1} \to \mathbb{E}^{\mu_2}$$

such that

(i) $q = (u, w) \in W^{\lambda_1}$ *iff* $w = \pi_{\lambda_2}(\psi_2)(u, \pi_{\mu_1}(u))$*, i.e.,*

$$W^{\lambda_1} = \mathrm{graph}(\phi_2) \ \textit{with}\ w = \phi_2(u) = \pi_{\lambda_2}(\psi_2)(u, \pi_{\mu_1}(u)).$$

(ii) $\mathcal{F}^\mu(q) = \mathrm{graph}(\psi_2(u, \cdot))$ *for* $q = (u, w) \in W^{\lambda_1}$*, i.e.,*

$$p = (x, y) \in \mathcal{F}^\mu(q) \ \textit{iff}\ y = \psi_2(u, x).$$

(iii) f *is Lipschitz on each* $\mathcal{F}^\mu(q)$ *and for an adapted norm the Lipschitz constant is* $\leq \alpha$ *uniformly for all* $q \in W^{\lambda_1}$*.*

(iv) $\mathcal{F}^\mu(\bar{q})$ *coincides with the* μ*-left manifold* $\mathcal{F}^\mu(\bar{q}) = W^{\mu_1}$ *and*

$$\mathbb{T}_{\bar{q}}\mathcal{F}^\mu(\bar{q}) \cong \mathbb{E}^{\mu_1}.$$

(v) If f *is* $C^{k,1}, k \geq 1$*,* ${\lambda_1}^k < \lambda_2$*,* ${\mu_1}^k < \mu_2$*, and* $\mu_1{\lambda_1}^k < \mu_2$*, then* ψ_2 *is* C^k*.*

(vi) $\mathcal{F}^\mu$ *is independent of any two different choices in* α*.*

The proof is an application of the Uniform Contraction Principle. The main idea is to construct the sub-foliation function ψ_2 as part of a fixed point of a uniform contraction map. We will break it up into a few lemmas. Before doing so, we first recall a few important properties about W^{λ_1} in the statements below from the proof for the Left-Manifold Theorem, assuming the fixed $\bar{q}$ is translated to $0 \in \mathbb{R}^d$.

Proposition 1. *For any $\lambda_1 < \beta < \lambda_2$, let S_β be a Banach space defined by*

$$S_\beta := \{\gamma = \{p_n\}_{n=0}^{\infty} : p_n \in \mathbb{R}^d,\ \sup\{\beta^{-n}\|p_n\| : n \geq 0\} < \infty\}$$

with norm

$$\|\gamma\|_\beta = \sup\{\beta^{-n}\|p_n\| : n \geq 0\}.$$

For any sufficiently small $\|f - Df(\bar{q})\|_1$, the orbit $\gamma_q = \{f^n(q)\}_{n=0}^{\infty}$ of any point $q = q_0 = (u, w) \in W^{\lambda_1}$ can be expressed as a function $\gamma_q = \gamma^*(u)$ for $u \in \mathbb{E}^{\lambda_1}$ so that $\gamma^* \in C^k(\mathbb{E}^{\lambda_1}, S_\beta)$ if $f \in C^k(\mathbb{R}^d)$ and ${\lambda_1}^k < \lambda_2$. Moreover, for any $u,\ u' \in \mathbb{E}^{\lambda_1}$

$$\|\gamma^*(u) - \gamma^*(u')\|_\beta \leq \tfrac{1}{1-\theta_\lambda(\beta)}\|u - u'\| \tag{1}$$

where $0 < \theta_\lambda(\beta) < 1$ is a uniform contraction constant depending on β. Furthermore, let $\phi_2(u)$ be the $\mathbb{E}^{\lambda_2}$-component of $\gamma^*(u)$'s initial point, then ϕ_2 is in $C^k(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$ and

$$W^{\lambda_1} = \operatorname{graph}(\phi_2),\ \ \phi_2(0) = 0, \ \ \textit{and}\ \ D\phi_2(0) = 0.$$

We also recall that by the Variation of Parameters Formula Theorem (VPF) for splitting $\mathbb{R}^d \cong \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ corresponding to $Df(\bar{q}) \cong \operatorname{diag}(A_1, A_2)$, the map $(\bar{x}, \bar{y}) = f(x, y)$ with $(x, y), (\bar{x}, \bar{y}) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ is equivalent to

$$\left\{\begin{array}{l} \bar{x} = A_1 x + h_1(x, y) \\ y = A_2^{-1}\bar{y} + h_2(\bar{x}, \bar{y}), \end{array}\right. \tag{2}$$

and for any orbit, $p_n = (x_n, y_n) = f(x_{n-1}, y_{n-1})$, and $n \geq 0$

$$\left\{\begin{array}{l} x_n = A_1^n x_0 + \sum_{i=1}^{n} A_1^{n-i} h_1(p_{i-1}) \\ y_n = A_2^{n-m} y_m + \sum_{i=n+1}^{m} A_2^{n+1-i} h_2(p_i). \end{array}\right. \tag{3}$$

Here, the functions $h_1,\ h_2$ are defined by f and are as smooth as f, satisfying

$$h_1(0) = 0,\ Dh_1(0) = 0,\ h_2(0) = 0,\ Dh_2(0) = 0. \tag{4}$$

They are globally Lipschitz and their Lipschitz constant can be taken to be

$$L = \|D(h_1, h_2)\|_0 \to 0 \ \text{ as } \ \|f - Df(\bar{q})\|_1 \to 0. \tag{5}$$

The result above holds for sufficiently small $\|f - Df(\bar{q})\|_1$.

Associated with h_i, we will need the following functions throughout

$$g_i(q, \delta p) = h_i(q + \delta p) - h_i(q), \ \text{ for } \ i = 1, 2. \tag{6}$$

Because $h_i \in C^{k,1}$ so is $g_i \in C^{k,1}$ satisfying

$$g_i(0,0)=0,\ D_p g_i(0,0)=0,\ D_{\delta p} g_i(0,0)=0,\ \text{for}\ \ i=1,2. \tag{7}$$

More importantly, all derivatives in q satisfy

$$D_q^j g_i(q,0)=0,\ \ \text{for}\ \ 0\le j\le k,\ \text{and}\ \ i=1,2. \tag{8}$$

To save notation, we will use the same notation for Lipschitz constants of both h_i and g_i

$$L=\max\{\|D(h_1,h_2)\|_0,\|D(g_1,g_2)\|_0\}\to 0\ \ \text{as}\ \ \|f-Df(\bar q)\|_1\to 0. \tag{9}$$

Since $g_i \in C^{k,1}$, we will denote by $L_1, L_2, \dots, L_k$ the Lipschitz constants for $D_q g_i, D_q^2 g_i, \dots, D_q^k g_i$, respectively. Together with the fact that $D_q^j g_i(q,0)=0$ we have

$$\|D_q^j g_i(q,\delta p)\| \le L_j \|\delta p\|\ \ \text{for}\ \ 0\le j\le k,\ \text{and}\ \ i=1,2. \tag{10}$$

Unlike L which can be made as small as possible by making $\|f-Df(\bar q)\|_1$ small, these constants L_j are not necessarily small.

We will repeatedly use this formula for geometric sequences

$$a+ar+ar^2+\cdots+ar^{n-1}=\tfrac{a(1-r^n)}{1-r},\ \text{for}\ r\neq 1$$

and its differentiation formulas in r. We will denote throughout

$$\gamma_p=\{p_n=f^n(p)\}_{n=0}^{\infty}$$

the orbit of f with the initial point p, for which $p_0=p$. The proof now consists of a sequence of lemmas below.

Lemma 1. *For any parameter α satisfying $\mu_1<\alpha<\mu_2$, let*

$$\Delta S_\alpha := \{\delta\gamma=\{\delta p_n\}_{n=0}^{\infty} : \delta p_n \in \mathbb{E}^{\mu_1}\times\mathbb{E}^{\mu_2}, \sup\{\alpha^{-n}\|\delta p_n\| : n\ge 0\}<\infty\} \tag{11}$$

with norm

$$\|\delta\gamma\|_\alpha=\sup\{\alpha^{-n}\|\delta p_n\| : n\ge 0\}.$$

For any $q=q_0=(u,w)\in W^{\lambda_1}$ with $\gamma_q=\{q_n\}$ and $\delta p=\{\delta p_n\}\in\Delta S_\alpha$, let $\overline{\delta\gamma}=T(\delta\gamma)$ be defined by the equations below

$$\begin{cases} \overline{\delta x}_n = A_1^n \delta x_0 + \sum_{i=1}^n A_1^{n-i} g_1(q_{i-1},\delta p_{i-1}) \\ \overline{\delta y}_n = \sum_{i=n+1}^{\infty} A_2^{n+1-i} g_2(q_i,\delta p_i) \end{cases} \tag{12}$$

Then $\overline{\delta\gamma}\in\Delta S_\alpha$. Specifically, let ν,η be any parameters satisfying

$$\mu_1<\nu<\alpha<1/\eta<\mu_2, \tag{13}$$

then an adapted norm can be chosen so that

$$\|\overline{\delta\gamma}\|_\alpha \le \|\delta x_0\| + \tfrac{L\|\delta\gamma\|_\alpha}{\alpha-\nu} + \tfrac{\alpha L\|\delta\gamma\|_\alpha}{1-\alpha\eta}. \tag{14}$$

More importantly, $p\in\mathcal{F}^\mu(q)$ iff the orbit difference $\delta\gamma=\gamma_p-\gamma_q$ is a fixed point of T, i.e., $p=q+\delta p$ with $\delta p=(\delta x_0,\delta y_0)$ the initial point of $\delta\gamma$, and specifically,

$$p=\pi_\mu(u,\phi_2(u))+\left(\delta x_0, \textstyle\sum_{i=1}^{\infty} A_2^{1-i} g_2(q_i,\delta p_i)\right). \tag{15}$$

Proof. We first show that T is well-defined together with the bound estimate. We begin by fixing an adapted norm for the condition (13) so that the following inequalities hold

$$\|A_1\| < \nu < \alpha < 1/\eta < \|A_2\| \text{ and } \|A_2^{-1}\| < \eta < \tfrac{1}{\alpha}. \tag{16}$$

We now demonstrate $\overline{\delta\gamma} = \{(\overline{\delta x}_n, \overline{\delta y}_n)\} \in \Delta S_\alpha$. Because $g_i(q,0) = 0$ and $\|g_i(q,\delta p)\| \le L\|\delta p\|$ from (8, 10) we have for $\overline{\delta x}_n$

$$\begin{aligned} \|\overline{\delta x}_n\| &\le \|A_1^n\|\|\delta x_0\| + \textstyle\sum_{i=1}^n \|A_1^{n-i} g_1(q_{i-1}, \delta p_{i-1})\| \\ &\le \nu^n\|\delta x_0\| + \textstyle\sum_{i=1}^n \nu^{n-i} L\alpha^{i-1}\|\delta\gamma\|_\alpha \\ &= \nu^n\|\delta x_0\| + L\|\delta\gamma\|_\alpha \tfrac{\alpha^n - \nu^n}{\alpha-\nu} \\ &\le (\|\delta x_0\| + \tfrac{L\|\delta\gamma\|_\alpha}{\alpha-\nu})\alpha^n. \end{aligned} \tag{17}$$

Similarly,

$$\begin{aligned} \|\overline{\delta y}_n\| &\le \textstyle\sum_{i=n+1}^\infty \|A_2^{n+1-i} g_2(q_i, \delta p_i)\| \\ &\le \textstyle\sum_{i=n+1}^\infty \eta^{i-n-1} L\alpha^i \|\delta\gamma\|_\alpha \\ &= \eta^{-n-1} L\|\delta\gamma\|_\alpha \tfrac{(\alpha\eta)^{n+1}}{1-\alpha\eta} \\ &= \tfrac{\alpha L\|\delta\gamma\|_\alpha}{1-\alpha\eta}\alpha^n. \end{aligned} \tag{18}$$

Hence, the estimate (14) holds. This shows that the infinite series converges uniformly and that T is well-defined, mapping ΔS_α into itself.

Next, we show the last part of the lemma. First, for $p \in \mathcal{F}^\mu(q)$, both orbits γ_p, γ_q are in S_β, and the orbit difference

$$\delta\gamma = \gamma_p - \gamma_q = \{\delta p_n : \delta p_n = p_n - q_n, n \ge 0\} \tag{19}$$

is in ΔS_α by definition. By the VPF (3), $\delta\gamma$ satisfies

$$\begin{cases} \delta x_n = A_1^n \delta x_0 + \sum_{i=1}^n A_1^{n-i} g_1(q_{i-1}, \delta p_{i-1}) \\ \delta y_n = A_2^{n-m}\delta y_m + \sum_{i=n+1}^m A_2^{n+1-i} g_2(q_i, \delta p_i) \end{cases}$$

Because $\|\delta y_m\| \le \alpha^m \|\delta\gamma\|_\alpha$ and $\|A_2^{n-m}\| \le \eta^{m-n}$ and $\alpha\eta < 1$, the first term in the y_n-equation above converges to 0 as $m \to \infty$. The estimate (18) also shows the partial sum of the y_n-equation converges uniformly. Therefore the limit as $m \to \infty$ exists for the y_n-equation and the limit is exactly the y_n-equation for the map T. Hence, $\delta\gamma$ is a fixed point of T.

Conversely, assume $\delta\gamma = \{(\delta x_n, \delta y_n)\}$ is a fixed point of T for a given γ_q from W^{λ_1}. It is straightforward to verify

$$\begin{cases} \delta x_n = A_1 \delta x_{n-1} + g_1(q_{n-1}, \delta p_{n-1}) \\ \delta y_n = A_2^{-1}\delta y_{n+1} + g_2(q_{n+1}, \delta p_{n+1}). \end{cases}$$

Denote $p_n = q_n + \delta p_n,\ p_n = (x_n, y_n),\ q_n = \pi_\mu(q_n) = (x_{q,n}, y_{q,n}) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$. Then because γ_q is an orbit it satisfies

$$\begin{cases} x_{q,n} = A_1 x_{q,n-1} + h_1(q_{n-1}) \\ y_{q,n} = A_2^{-1} y_{q,n+1} + h_2(q_{n+1}). \end{cases}$$

Sum up these two equations component by component to obtain

$$\begin{cases} x_n = A_1 x_{n-1} + h_1(p_{n-1}) \\ y_n = A_2^{-1} y_{n+1} + h_2(p_{n+1}), \end{cases}$$

which shows $\gamma_p = \{p_n\} = \gamma_q + \delta\gamma$ must be an orbit of f. Since $\gamma_q \in S_\beta$ and $\delta\gamma \in \Delta S_\alpha \subset S_\beta$, we also have $\gamma_p \in S_\beta$. Hence, the initial point, p_0, of γ_p is in W^{λ_1} and in $\mathcal{F}^\mu(q)$ by definition. Last, the identity (15) follows by writing out the initial point of γ_p because $q = (u, \phi_2(u)) \in W^{\lambda_1}$. □

Lemma 2. *Let $\phi_2 \in C^1(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$ be the function whose graph is W^{λ_1}. Then there is a function $\psi_2 : \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1} \to \mathbb{E}^{\mu_2}$ so that for all $u \in \mathbb{E}^{\lambda_1}$,*

$$\phi_2(u) = \pi_{\lambda_2}(\psi_2)(u, \pi_{\mu_1}(u))$$

and for every $q \in W^{\lambda_1}$ with $q = (u, \phi_2(u))$

$$\mathcal{F}^\mu(u) := \mathcal{F}^\mu(q) = \text{graph}(\psi_2(u, \cdot)). \tag{20}$$

Moreover, the definition of $\mathcal{F}^\mu$ is independent of any two different choices in α.

Proof. By Lemma 1, $p \in \mathcal{F}^\mu(q)$ iff $p = q + \delta p$ with $\delta p = \delta p_0$ the initial point of a fixed point $\delta\gamma = \{\delta p_n\}_{n\geq 0}$ of the map T from its proof. We already know from Proposition 1 that q is parameterized by $u \in \mathbb{E}^{\lambda_1}$ by $q = (u, \phi_2(u))$ as well as its orbit $\gamma_q = \gamma^*(u) = \{q_n(u)\}$. We only need to show $\delta p = (\delta x_0, \delta y_0)$ exists and is parameterized by u and by its $\mathbb{E}^{\mu_1}$-coordinate δx_0 which we will replace by $\delta x = \delta x_0 \in \mathbb{E}^{\mu_1}$. In fact, if that is true, then in this parameterized designation, the function ψ_2 must be defined from the identity (15) as below

$$p = (x, y) = (x, \psi_2(u, x)) := \pi_\mu(u, \phi_2(u)) + (\delta x, \delta y_0(u, \delta x))$$

where $\delta x = x - \pi_{\mu_1}(q) = x - \pi_{\mu_1}(u)$, namely

$$\psi_2(u, x) = \pi_{\mu_2}(u, \phi_2(u)) + \textstyle\sum_{i=1}^\infty A_2^{1-i} g_2(q_i(u), \delta p_i(u, x - \pi_{\mu_1}(u))) \, . \tag{21}$$

Assuming the fixed point $\delta\gamma$ is unique for T, then we see the zero sequence $\delta\gamma = \{0\}$ is a trivial fixed point if $\delta x = 0$. As a consequence, we get

$$\psi_2(u, \pi_{\mu_1}(u)) = \pi_{\mu_2}(u, \phi_2(u)) + \delta y_0(u, 0) = \pi_{\mu_2}(u, \phi_2(u))$$

which gives

$$\pi_{\lambda_2}(\psi_2)(u, \pi_{\mu_1}(u)) = \phi_2(u),$$

the inclusion of W^{λ_1}. Definition (21) obviously shows (20). Therefore, it is only left to show the existence and uniqueness of fixed point of T for each u, δx, and their independence on any two choices in α.

To this end, we will consider T as a parameterized map $T : \Delta S_\alpha \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1} \to \Delta S_\alpha$ with $\overline{\delta\gamma} = T(\delta\gamma, u, \delta x)$ being defined by (12) as below

$$\begin{cases} \overline{\delta x}_n = A_1^n \delta x + \sum_{i=1}^n A_1^{n-i} g_1(q_{i-1}(u), \delta p_{i-1}) \\ \overline{\delta y}_n = \sum_{i=n+1}^\infty A_2^{n+1-i} g_2(q_i(u), \delta p_i) \end{cases} \tag{22}$$

We first show T is a uniform contraction. By the proof of Lemma 1, $T(\cdot, u, \delta x)$ maps ΔS_α into ΔS_α. For its uniform contraction, let $\delta\gamma, \delta\gamma' \in \Delta S_\alpha$ and $\overline{\delta\gamma} = T(\delta\gamma, u, \delta x), \overline{\delta\gamma}\,' = T(\delta\gamma', u, \delta x)$. Then we have

$$\begin{aligned} \|\overline{\delta x}_n - \overline{\delta x}_n\,'\| &\leq \textstyle\sum_{i=1}^n \|A_1^{n-i}[g_1(q_{i-1}(u), \delta p_{i-1}) - g_1(q_{i-1}(u), \delta p'_{i-1})]\| \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} L \|\delta p_{i-1} - \delta p'_{i-1}\| \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} L \alpha^{i-1} \|\delta\gamma - \delta\gamma'\|_\alpha \\ &\leq \tfrac{L}{\alpha-\nu} \alpha^n \|\delta\gamma - \delta\gamma'\|_\alpha \end{aligned} \tag{23}$$

and

$$\begin{aligned} \|\overline{\delta y}_n - \overline{\delta y}_n\,'\| &\leq \textstyle\sum_{i=n+1}^\infty \|A_2^{n+1-i}[g_2(q_i(u), \delta p_i) - g_2(q_i(u), \delta p'_i)]\| \\ &\leq \textstyle\sum_{i=n+1}^\infty \eta^{i-n-1} L \|\delta p_i - \delta p'_i\| \\ &\leq \textstyle\sum_{i=n+1}^\infty \eta^{i-n-1} \alpha^i \|\delta\gamma - \delta\gamma'\|_\alpha \\ &\leq \tfrac{L\alpha}{1-\alpha\eta} \alpha^n \|\delta\gamma - \delta\gamma'\|_\alpha . \end{aligned} \tag{24}$$

Hence,

$$\|T(\delta\gamma, u, \delta x) - T(\delta\gamma', u, \delta x)\|_\alpha \leq (\tfrac{L}{\alpha-\nu} + \tfrac{L\alpha}{1-\alpha\eta}) \|\delta\gamma - \delta\gamma'\|_\alpha$$

showing $T(\cdot, u, \delta x)$ is a uniform contraction in ΔS_α provided

$$\theta := \theta(\alpha) = \tfrac{L}{\alpha-\nu} + \tfrac{L\alpha}{1-\alpha\eta} < 1 \tag{25}$$

which is true for sufficiently small $\|f - Df(\bar{q})\|_1$. We denote the fixed point by

$$\delta\gamma^*(u, \delta x) = \{\delta p_n(u, \delta x) = (\delta x_n(u, \delta x), \delta y_n(u, \delta x))\}_{n=0}^\infty \tag{26}$$

Notice that the existence and uniqueness proof of $\delta\gamma^*$ above shows that for any

$$\mu_1 < \alpha' < \alpha < \mu_2 \quad \text{with } \|A_1\| < \nu < \alpha' < \alpha < 1/\eta < \|A_2\|, \|A_2^{-1}\| < \eta$$

as long as

$$\theta(\alpha'),\ \theta(\alpha) < 1$$

$T(\cdot, u, \delta x)$ has a unique fixed point in $\Delta S_{\alpha'}$ and ΔS_α. But since $\Delta S_{\alpha'}$ is a closed subspace of ΔS_α, the unique fixed point $\delta\gamma^*(u, \delta x)$ is in both $\Delta S_{\alpha'}$ and ΔS_α. This shows the independence of $\mathcal{F}^\mu$ on any two choices in α. □

Lemma 3. *The foliation function ψ_2 is $C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \mathbb{E}^{\mu_2})$.*

Proof. Notice from its definition (21) that we only need to show δp_0 is C^1 for which it suffices to show the unique fixed point $\delta\gamma^*$ of T from Lemma 2 is C^1 since δp_0 is only a point of the sequence $\delta\gamma^*$. By the Uniform Contraction Principle II, we need to verify two conditions: (1) $T(\delta\gamma, u, \delta x)$ is differentiable in $\delta\gamma$ and $\|D_{\delta\gamma} T(\delta\gamma, u, \delta x)\|$ is uniformly bounded by a constant smaller than 1; (2) $T \in C^1(\Delta S_\alpha \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$.

To show (1), let $\delta\gamma = \{\delta p_n\}, v = \{v_n\} \in \Delta S_\alpha$, and formally differentiate (22). Then $D_{\delta\gamma}T(\delta\gamma, u, \delta x)v$ needs to be as below in components:

$$\left\{ \begin{array}{l} [D_{\delta\gamma}T(\delta\gamma, u, \delta x)v]_{n,\,1} = \sum_{i=1}^{n} A_1^{n-i} D_{\delta p} g_1(q_{i-1}(u), \delta p_{i-1}) v_{i-1} \\ [D_{\delta\gamma}T(\delta\gamma, u, \delta x)v]_{n,\,2} = \sum_{i=n+1}^{\infty} A_2^{n+1-i} D_{\delta p} g_2(q_i(u), \delta p_i) v_i. \end{array} \right. \tag{27}$$

By the exactly same estimate as for (23) we have

$$\|[D_{\delta\gamma}T(\delta\gamma, u, \delta x)v]_{n,\,1}\| \le \tfrac{L}{\alpha-\nu}\alpha^n \|v\|_\alpha.$$

Similarly, by the exactly same estimate as for (24) we have

$$\|[D_{\delta\gamma}T(\delta\gamma, u, \delta x)v]_{n,\,2}\| \le \tfrac{L\alpha}{1-\alpha\eta}\alpha^n \|v\|_\alpha.$$

These estimates imply two conclusions. One, because of the uniform convergence of the second equation, the derivative $D_{\delta\gamma}T(\delta\gamma, \delta x)$ exists. Two, it shows the derivative is a bounded linear map in $L(\Delta S_\alpha, \Delta S_\alpha)$ whose α-norm

$$\|D_{\delta\gamma}T(\delta\gamma, u, \delta x)\|_\alpha \le \theta(\alpha) < 1,$$

is bounded by the same uniform contraction constant $\theta(\alpha)$ from (25).

To show (2), we separate it into three cases. The first case is done above for derivative in $\delta\gamma$, the second case is for derivative in δx, and the third case is for derivative in u. For the second case we have formally

$$[D_{\delta x}T(\delta\gamma, u, \delta x)]_{n,\,1} = A_1^n, \;\text{ and }\; [D_{\delta x}T(\delta\gamma, u, \delta x)]_{n,\,2} = 0,$$

which implies

$$\|[D_{\delta x}T(\delta\gamma, u, \delta x)]_n\| \le \|A_1^n\| \le \alpha^n$$

and $\|D_{\delta x}T(\delta\gamma, u, \delta x)\|_\alpha \le 1$ follows. That is, T is C^1 in δx.

To show the third case, we will treat T in u as a composition

$$T(\delta\gamma, u, \delta x) = \bar{T}(\delta\gamma, \gamma^*(u), \delta x) \tag{28}$$

of a map $\bar{T} : \Delta S_\alpha \times S_\beta \times \mathbb{E}^{\mu_1} \to \Delta S_\alpha$ with a C^1 map $\gamma^* : \mathbb{E}^{\lambda_1} \to S_\beta$ which is the orbit sequence $\gamma^*(u) = \gamma_q$ for point $q = (u, w) \in W^{\lambda_1}$. Here, $\overline{\delta\gamma} = \bar{T}(\delta\gamma, \gamma, \delta x)$ is defined as below

$$\left\{ \begin{array}{l} \overline{\delta x}_n = A_1^n \delta x + \sum_{i=1}^{n} A_1^{n-i} g_1(q_{i-1}, \delta p_{i-1}) \\ \overline{\delta y}_n = \sum_{i=n+1}^{\infty} A_2^{n+1-i} g_2(q_i, \delta p_i), \end{array} \right. \tag{29}$$

the same definition as T except for a general $\gamma = \{q_n\}_{n=0}^{\infty} \in S_\beta$. Because of the composition we only need to show $\bar{T}$ is C^1 in γ.

To this end, by the sub-tight split definition that $\mu_1\lambda_1 < \mu_2$, we can choose a parameter ς close to μ_1, α close to μ_2, and β close to λ_1 so that

$$\varsigma\beta < \alpha \quad \text{where} \quad \mu_1 < \varsigma < \alpha < \mu_2 \le \lambda_1 < \beta < \lambda_2, \tag{30}$$

for which an adjusted norm can be chosen to satisfy (16) and

$$\|A_1\| < \nu < \varsigma < \alpha < \|A_2\|. \tag{31}$$

We will treat the fixed point $\delta\gamma^*(u, \delta x)$ in both ΔS_ς and ΔS_α. We also use the estimates below from (10) that for $i = 1, 2$,

$$\|g_i(q, \delta p) - g_i(q', \delta p)\| \le \|D_q g_i(\cdot, \delta p)\|_0 \|q - q'\| \le L_1 \|\delta p\| \|q - q'\|. \tag{32}$$

Now, formally differentiating $\bar{T}$ in γ, we obtain from (29) any $v \in S_\beta$

$$\begin{cases} [D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,\,1} = \sum_{i=1}^n A_1^{n-i} D_q g_1(q_{i-1}, \delta p_{i-1}) v_{i-1} \\ [D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,\,2} = \sum_{i=n+1}^\infty A_2^{n+1-i} D_q g_2(q_i, \delta p_i) v_i, \end{cases} \tag{33}$$

Because of $\varsigma\beta < \alpha$, we have

$$\begin{aligned} \|[D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,\,1}\| &\le \textstyle\sum_{i=1}^n \|A_1^{n-i}\| L_1 \|\delta p_{i-1}\| \|v_{i-1}\| \\ &\le L_1 \textstyle\sum_{i=1}^n \nu^{n-i} \varsigma^{i-1} \|\delta\gamma\|_\varsigma \beta^{i-1} \|v\|_\beta \\ &\le L_1 \|\delta\gamma\|_\alpha \textstyle\sum_{i=1}^n \nu^{n-i} \alpha^{i-1} \|v\|_\beta \\ &\le \tfrac{L_1 \|\delta\gamma\|_\alpha}{\alpha - \nu} \alpha^n \|v\|_\beta. \end{aligned} \tag{34}$$

Similarly, we have

$$\begin{aligned} \|[D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,\,2}\| &\le \textstyle\sum_{i=n+1}^\infty \|A_2^{n+1-i}\| L_1 \|\delta p_i\| \|v_i\| \\ &\le L_1 \textstyle\sum_{i=n+1}^\infty \eta^{i-n-1} \varsigma^i \|\delta\gamma\|_\varsigma \beta^i \|v\|_\beta \\ &\le L_1 \|\delta\gamma\|_\alpha \textstyle\sum_{i=n+1}^\infty \eta^{i-n-1} \alpha^i \|v\|_\beta \\ &\le \tfrac{L_1 \|\delta\gamma\|_\alpha \alpha}{1 - \alpha\eta} \alpha^n \|v\|_\beta. \end{aligned} \tag{35}$$

Combine these two estimates to obtain

$$\|[D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)]\|_\alpha \le (\tfrac{1}{\alpha - \nu} + \tfrac{\alpha}{1 - \alpha\eta}) L_1 \|\delta\gamma\|_\alpha.$$

The convergence of the infinite series also shows the derivative exists. Hence $\bar{T}(\delta\gamma, \cdot, \delta x)$ is in $C^1(S_\beta, \Delta S_\alpha)$ and T is C^1 in u as needed. □

Lemma 4. *f is Lipschitz on $\mathcal{F}^\mu(q)$ and for the adapted norm from Lemma 1 the Lipschitz constant is $\le \alpha$ uniformly for all $q \in W^{\lambda_1}$.*

Proof. We remark first that since $T(\delta\gamma, u, \delta x)$ is Lipschitz continuous in δx with

$$\|T(\delta\gamma, u, \delta x) - T(\delta\gamma, u, \delta x')\|_\alpha \le \|\delta x - \delta x'\|,$$

because $\|A_1^n\| < \nu^n$, we have by the Uniform Contraction Principle I that $\delta\gamma^*$ satisfies

$$\|\delta\gamma^*(u, \delta x) - \delta\gamma^*(u, \delta x')\|_\alpha \le \tfrac{1}{1-\theta} \|\delta x - \delta x'\|. \tag{36}$$

For the proof of the lemma, we need to show that for any $q \in W^{\lambda_1}$ and for any $p, p' \in \mathcal{F}^\mu(q)$, $\|f(p) - f(p')\| \le \alpha \|p - p'\|$. Let γ_p, $\gamma_{p'}$ be the orbits through p, p', respectively. Then $\delta\gamma^* = \gamma_p - \gamma_q$ and $\delta\gamma^{*\prime} = \gamma_{p'} - \gamma_q$ are fixed points of

$T(\cdot, u, \delta x)$ and $T(\cdot, u, \delta x')$, respectively, with $\delta x = x - \pi_{\mu_1}(u)$, $\delta x' = x' - \pi_{\mu_1}(u)$, $\delta\gamma^* = \delta\gamma^*(u, \delta x)$, and $\delta\gamma^{*\prime} = \delta\gamma^*(u, \delta x')$. More importantly,

$$\gamma_p - \gamma_{p'} = (\gamma_p - \gamma_q) - (\gamma_{p'} - \gamma_q) = \delta\gamma^* - \delta\gamma^{*\prime}$$

whose second point on the sequence is

$$f(p) - f(p') = p_1 - p_1' = \delta p_1(u, \delta x) - \delta p_1(u, \delta x').$$

The $\mathbb{E}^{\mu_1}$-coordinate of the right side can be estimated as

$$\begin{aligned}
\|\overline{\delta x}_1 - \overline{\delta x}_1{}'\| &\leq \|A_1(\delta x - \delta x') + g_1(q_0(u), \delta x) - g_1(q_0(u), \delta x')\| \\
&\leq \nu\|\delta x - \delta x'\| + \|h_1(q_0(u) + \delta x) - h_1(q_0(u) + \delta x')\| \\
&\leq \nu\|\delta x - \delta x'\| + L\|\delta x - \delta x'\| \\
&\leq (\nu + L)\|p - p'\|
\end{aligned}$$

The $\mathbb{E}^{\mu_2}$-coordinate of the right side is

$$\begin{aligned}
\|\overline{\delta y}_1 - \overline{\delta y}_1{}'\| &\leq \textstyle\sum_{i=2}^{\infty} \|A_2^{2-i}[g_2(q_i(u), \delta p_i) - g_2(q_i(u), \delta p_i')]\| \\
&= \textstyle\sum_{i=2}^{\infty} \|A_2^{2-i}[h_2(q_i(u) + \delta p_i) - h_2(q_i(u) + \delta p_i')]\| \\
&\leq \textstyle\sum_{i=2}^{\infty} \eta^{i-2} L\alpha^i \|\delta\gamma^* - \delta\gamma^{*\prime}\|_\alpha \\
&\leq \tfrac{L\alpha^2}{1-\alpha\eta}\|\delta\gamma^*(u, \delta x) - \delta\gamma^*(u, \delta x')\|_\alpha \\
&\leq \tfrac{L\alpha^2}{1-\alpha\eta}\tfrac{1}{1-\theta}\|\delta x - \delta x'\| \\
&\leq \tfrac{L\alpha^2}{1-\alpha\eta}\tfrac{1}{1-\theta}\|p - p'\|
\end{aligned}$$

where (36) is used for the second last estimate. Therefore,

$$\|f(p) - f(p')\| \leq (\nu + L + \tfrac{L\alpha^2}{1-\alpha\eta}\tfrac{1}{1-\theta})\|p - p'\| \leq \alpha\|p - p'\|$$

for sufficiently small L, i.e., for sufficiently small $\|f - Df(\bar{q})\|_1$ since $\nu < \alpha$. □

Lemma 5. *If f is $C^{k,1}$, ${\lambda_1}^k < \lambda_2$, ${\mu_1}^k < \mu_2$, and $\mu_1{\lambda_1}^k < \mu_2$, then ψ_2 is C^k.*

Proof. The case of $k = 1$ is done in Lemma 3. For the case of $k \geq 2$, we notice first from the definition of ψ_2, (21), that we need to prove both ϕ_2 and $\delta p_0(u, \delta x)$ are C^k. By the λ-left manifold theorem we know from Proposition 1 that ϕ_2 is C^k because ${\lambda_1}^k < \lambda_2$. Hence, we only need to show $\delta p_0(u, \delta x)$ is C^k, which in turn suffices to show the fixed point function $\delta\gamma^*(u, \delta x)$ is C^k. Similar to the proof for the λ-left manifold theorem, the Uniform Contraction Principle II cannot be applied directly for $k = 2$ (except for the special case where $\mu_1 \leq 1$). This is because we cannot prove in general $T \in C^k(\Delta S_\alpha \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$. An indirect approach is needed.

First, we want to prove instead the following claim

$$T \in C^k(\Delta S_\varsigma \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha) \tag{37}$$

where ς is sufficiently close to μ_1 and α is sufficiently close to μ_2 satisfying

$$\mu_1 < \varsigma < \alpha < \mu_2 \quad \text{and} \quad {\mu_1}^k < \varsigma^k < \alpha < \mu_2 \tag{38}$$

which is guaranteed by the assumption ${\mu_1}^k < \mu_2$.

To prove the claim, we first fix an adapted norm for $(x, y) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ so that in addition to (16) we also have

$$\|A_1\| < \nu < \varsigma < \alpha < \|A_2\|. \tag{39}$$

We separate the proof into four cases. The first case is for derivatives in δx, the second case is for derivatives in $\delta\gamma$, the third case is for derivatives in u, and the fourth case is about mixed derivatives. For the first case we already have from the proof of Lemma 3 that

$$[D_{\delta x} T(\delta\gamma, u, \delta x)]_{n,\,1} = A_1^n, \text{ and } [D_{\delta x} T(\delta\gamma, u, \delta x)]_{n,\,2} = 0,$$

i.e., T is C^1 in δx. And $D^j_{\delta x} T(\delta\gamma, u, \delta x) = 0$, for $2 \le j \le k$, the zero operators. Hence, it is obvious that $T(\delta\gamma, u, \cdot) \in C^k(\mathbb{E}^{\mu_1}, \Delta S_\alpha)$ for $\delta\gamma \in \Delta S_\varsigma$.

For the second case, applying the case of $j = 1$ for $\mu_1 < \varsigma < \alpha < \mu_2$, we know $D_{\delta\gamma} T(\delta\gamma, u, \delta x) \in L(\Delta S_\varsigma, \Delta S_\varsigma) \subset L(\Delta S_\varsigma, \Delta S_\alpha)$ because $\Delta S_\varsigma \subset \Delta S_\alpha$ holds automatically. For any $2 \le j \le k$, $[D^j_{\delta\gamma} T(\delta\gamma, u, \delta x)]$ needs to be a j-linear form ΔS_ς to ΔS_α. To this end, let $v = v^1 \otimes v^2 \otimes \cdots \otimes v^j$ with each $v^\ell \in \Delta S_\varsigma,\ 1 \le \ell \le j$. Formally differentiate (22) to get

$$\begin{cases} [D^j_{\delta\gamma} T(\delta\gamma, u, \delta x)v]_{n,\,1} = \sum_{i=1}^n A_1^{n-i} D^j_{\delta p} g_1(q_{i-1}(u), \delta p_{i-1}) v_{i-1} \\ [D^j_{\delta\gamma} T(\delta\gamma, u, \delta x)v]_{n,\,2} = \sum_{i=n+1}^\infty A_2^{n+1-i} D^j_{\delta p} g_2(q_i(u), \delta p_i) v_i, \end{cases} \tag{40}$$

where

$$v_i = v_i^1 \otimes v_i^2 \otimes \cdots \otimes v_i^j, \quad v_i^\ell \in \mathbb{R}^d.$$

Similar to the estimate of (23) and because $g_i(q, \delta p) = h_i(q + \delta p) - h_i(q)$, $\varsigma < \alpha$, and $\varsigma^k < \alpha$, we have

$$\begin{aligned} \|[D^j_{\delta\gamma} T(\delta\gamma, u, \delta x)v]_{n,\,1}\| &\le \textstyle\sum_{i=1}^n \|A_1^{n-i}\| \|D^j h_1\| \|v_{i-1}\| \\ &\le \textstyle\sum_{i=1}^n \nu^{n-i} \|h_1\|_j \Pi_{\ell=1}^j \|v_{i-1}^\ell\| \\ &\le \|h_1\|_k \textstyle\sum_{i=1}^n \nu^{n-i} \varsigma^{j(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \\ &\le \|h_1\|_k \textstyle\sum_{i=1}^n \nu^{n-i} \alpha^{(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\le \tfrac{\|h_1\|_k}{\alpha-\nu} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\alpha. \end{aligned} \tag{41}$$

Here we used the property that $\varsigma < \alpha$ and $\varsigma^k < \alpha$ imply $\varsigma^j < \alpha$ for $1 \le j \le k$. Similar to (24) we have

$$\begin{aligned} \|[D^j_{\delta\gamma} T(\delta\gamma, u, \delta x)v]_{n,\,2}\| &\le \textstyle\sum_{i=n+1}^\infty \|A_2^{n+1-i}\| \|D^j h_2\| \|v_i\| \\ &\le \textstyle\sum_{i=n+1}^\infty \eta^{i-n-1} \|h_2\|_j \varsigma^{ji} \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \\ &\le \|h_2\|_k \eta^{-n-1} \textstyle\sum_{i=n+1}^\infty (\eta\alpha)^i \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\le \tfrac{\|h_2\|_k \alpha}{1-\alpha\eta} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\alpha. \end{aligned} \tag{42}$$

Combine these two estimates to obtain

$$\|[D^j_{\delta\gamma} T(\delta\gamma, u, \delta x)]\|_\alpha \le \|(h_1, h_2)\|_k \max\{\tfrac{1}{\alpha-\nu}, \tfrac{\alpha}{1-\alpha\eta}\}.$$

The convergence of the infinite series also shows the derivatives are well-defined. This completes the proof that $T(\cdot, u, \delta x) \in C^k(\Delta S_\varsigma, \Delta S_\alpha)$.

For the third case $T(\delta\gamma, \cdot, \delta x) \in C^k(\mathbb{E}^{\lambda_1}, \Delta S_\alpha)$, we will treat it similarly as in the proof of Lemma 3 to be a composition of $\bar{T}(\delta\gamma, \cdot, \delta x) : S_\beta \to \Delta S_\alpha$, defined by (28, 29), with $\gamma = \gamma^* : \mathbb{E}^{\lambda_1} \to S_\beta$ for each $\delta\gamma \in \Delta S_\varsigma$. By Proposition 1, $\gamma^* \in C^k(\mathbb{E}^{\lambda_1}, S_\beta)$. So we only need to show $\bar{T}(\delta\gamma, \cdot, \delta x)$ is C^k. The $k = 1$ case is treated in the proof of Lemma 3.

For $k \geq 2$, we use the condition that $\mu_1 {\lambda_1}^k < \mu_2$ to fix parameter ς close to μ_1, β close to λ_1, and α close to μ_2 so that in addition to (38) we also have

$$\varsigma\beta^k < \alpha \quad \text{where} \quad \mu_1 < \varsigma < \alpha < \mu_2 \leq \lambda_1 < \beta < \lambda_2. \tag{43}$$

As a result we assume that an adapted norm is chosen so that both (16) and (39) hold. We will also use the property (10) that

$$\|D_q^j g_i(q, \delta p)\| \leq \bar{L}\|\delta p\| \;\text{ with } \bar{L} = \max\{L_j : 1 \leq j \leq k\}. \tag{44}$$

We are now ready to show $\bar{T}(\delta\gamma, \cdot, \delta x) \in C^k(S_\beta, \Delta S_\alpha)$ for $\delta\gamma \in \Delta S_\varsigma$ $\delta x \in \mathbb{E}^{\mu_1}$. We need to show $[D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)]$ is a bounded j-linear form from S_β to ΔS_α. To this end, let $v = v^1 \otimes v^2 \otimes \cdots \otimes v^j$ with each $v^\ell \in S_\beta,\ 1 \leq \ell \leq j$. Formally differentiate (29) in $\gamma \in S_\beta$ to get

$$\begin{cases} [D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,\,1} = \sum_{i=1}^n A_1^{n-i} D_q^j g_1(q_{i-1}, \delta p_{i-1}) v_{i-1} \\ [D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,\,2} = \sum_{i=n+1}^\infty A_2^{n+1-i} D_q^j g_2(q_i, \delta p_i) v_i, \end{cases} \tag{45}$$

where

$$v_i = v_i^1 \otimes v_i^2 \otimes \cdots \otimes v_i^j, \quad v_i^\ell \in \mathbb{R}^d.$$

Similar to the estimate of (34) and because of (44), we have

$$\begin{aligned} \|[D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,\,1}\| &\leq \textstyle\sum_{i=1}^n \|A_1^{n-i}\| L_j \|\delta p_{i-1}\| \|v_{i-1}\| \\ &\leq \bar{L} \textstyle\sum_{i=1}^n \nu^{n-i} \varsigma^{i-1} \|\delta\gamma\|_\varsigma \Pi_{\ell=1}^j \|v_{i-1}^\ell\| \\ &\leq \bar{L}\|\delta\gamma\|_\alpha \textstyle\sum_{i=1}^n \nu^{n-i}\varsigma^{i-1}\beta^{j(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\beta \\ &\leq \bar{L}\|\delta\gamma\|_\alpha \textstyle\sum_{i=1}^n \nu^{n-i}\alpha^{i-1} \Pi_{\ell=1}^j \|v^\ell\|_\beta \\ &\leq \tfrac{\bar{L}\|\delta\gamma\|_\alpha}{\alpha-\nu} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\beta, \end{aligned}$$

where $\varsigma\beta^j < \alpha, 1 \leq j \leq k$ because $\varsigma < \alpha$ and $\varsigma\beta^k < \alpha$. Similar to the estimate of (35) we have

$$\begin{aligned} \|[D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,\,2}\| &\leq \textstyle\sum_{i=n+1}^\infty \|A_2^{n+1-i}\| L_j \|\delta p_i\| \|v_i\| \\ &\leq \bar{L} \textstyle\sum_{i=n+1}^\infty \eta^{i-n-1} \varsigma^i \|\delta\gamma\|_\varsigma \beta^{ji} \Pi_{\ell=1}^j \|v^\ell\|_\beta \\ &\leq \bar{L}\|\delta\gamma\|_\alpha \textstyle\sum_{i=n+1}^\infty \eta^{i-n-1}\alpha^i \Pi_{\ell=1}^j \|v^\ell\|_\beta \\ &\leq \tfrac{\bar{L}\|\delta\gamma\|_\alpha \alpha}{1-\alpha\eta} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\beta. \end{aligned}$$

Combine these two estimates to obtain

$$\|[D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)]\|_\alpha \leq (\tfrac{1}{\alpha-\nu} + \tfrac{\alpha}{1-\alpha\eta})\bar{L}\|\delta\gamma\|_\alpha.$$

The convergence of the infinite series also shows the derivatives are well-defined. Hence $\bar{T}(\delta\gamma, \cdot, \delta x)$ is in $C^k(S_\beta, \Delta S_\alpha)$, showing T is C^k in u.

For the fourth case about mixed derivatives of T in all variables, the arguments above for $\delta\gamma, u, \delta x$ can be combined to show all derivatives up to order k exist for T. This completes the proof for the claim (37).

We are now ready to show $\delta\gamma^*(\cdot) \in C^k(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$. By the Uniform Contraction Principle II for $T \in C^1(\Delta S_\varsigma \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\varsigma)$, the fixed point $\delta\gamma^*(\cdot)$ is in $C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\varsigma)$ and its derivative in $(u, \delta x)$ is given by

$$D\delta\gamma^*(\cdot) = \textstyle\sum_{n=0}^{\infty}[D_\gamma T(\delta\gamma^*(\cdot), \cdot)]^n D_{(u,\delta x)} T(\delta\gamma^*(\cdot), \cdot).$$

Since $\delta\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\varsigma)$, $T \in C^1(\Delta S_\varsigma \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\varsigma) \subset C^1(\Delta S_\varsigma \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, and $T \in C^k(\Delta S_\varsigma \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, $k \geq 2$ by the claim, here is the key to notice that the composition $D_\gamma T(\delta\gamma^*(\cdot), \cdot)$ is $C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$. This implies that the infinite series on the right is in $C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, and therefore, $D\delta\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, and $\delta\gamma^*(\cdot) \in C^2(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$ follows. Apply this argument recursively to obtain $\delta\gamma^*(\cdot) \in C^3(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, and so on until we reach $\delta\gamma^*(\cdot) \in C^k(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$. As a component of the initial point of $\delta\gamma^*$, δp_0 is in $C^k(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \mathbb{E}^{\mu_2})$, completing the proof that ψ_2 is C^k. □

Proof of Theorem 1. After the preceding lemmas, it only remains to point out that by the definition of W^{μ_1}, which is an invariant subspace of W^{λ_1}, it coincides with the definition of the foliation through the fixed point, $\mathcal{F}^\mu(\bar{q})$, i.e., $W^{\mu_1} = \mathcal{F}^\mu(\bar{q})$. In fact, we can show the tangent space directly as below. Since $\bar{q} \sim u = 0$ and $\phi_2(0) = 0$, we have from (21)

$$\psi_2(0, x) = \delta y_0(0, x) = \textstyle\sum_{i=1}^{\infty} A_2^{1-i} h_2(\delta p_i(0, x))$$

whose partial derivative in $x \in \mathbb{E}^{\mu_1}$ at the fixed point $\bar{q} \sim x = 0$ is

$$D_x\psi_2(0, 0) = \textstyle\sum_{i=1}^{\infty} A_2^{1-i} Dh_2(\delta p_i(0, 0)) D_x\delta p_i(0, 0) = 0$$

since $\delta\gamma^*(0, 0) = \{0\}$ and $Dh_i(0) = 0$, showing $\mathbb{T}_{\bar{q}}\mathcal{F}^\mu(0) = \mathbb{E}^{\mu_1}$. □

Remark: We can see from the proofs above that if the λ-left manifold point q is fixed at the fixed point $\bar{q}$ throughout, then the extra Lipschitz continuity condition for the highest derivative of f is not needed. That is, the μ-left manifold $\mathcal{F}^\mu(\bar{q}) = W^{\mu_1}$ is C^k if f is C^k plus ${\mu_1}^k < \mu_2$. This is because in this case, $g(\bar{q}, \delta p) = h(p)$ with $\bar{q} = 0,\ \delta p = p$.