# Center Manifold Theorem

Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism $f$ in $\mathbb{R}^d$. Let $J = Df(\bar{q})$, and denote

$$\sigma^s = \sigma(J) \cap \{|z| < 1\}, \sigma^c = \sigma(J) \cap \{|z| = 1\}, \text{ and } \sigma^u = \sigma(J) \cap \{|z| > 1\}$$

the set of stable eigenvalues, center eigenvalues, unstable eigenvalues, respectively, of the linearizatoin $Df(\bar{q})$. Let

$$\sigma^{cs} = \sigma^s \cup \sigma^c, \text{ and } \sigma^{cu} = \sigma^c \cup \sigma^u.$$

**Definition 1.** *Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism $f$ in $\mathbb{R}^d$ and $\beta$ be any constant satisfying*

$$1 < \beta < \min\{|\sigma^u|\}.$$

*The center-stable manifold of the fixed point $\bar{q}$ for $f$ is*

$$W^{\text{cs}} = \{p : \{\beta^{-n}[f^n(p) - \bar{q}]\}_{n=0}^{\infty} \textit{ is a bounded sequence}\}.$$

**Theorem 1** (Center-Stable Manifold Theorem)**.** *Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism $f$ in $\mathbb{R}^d$ with splitting $\mathbb{R}^d \cong \mathbb{E}^{cs} \times \mathbb{E}^u$. Then a sufficiently small $\|f - Df(\bar{q})\|_1$ implies $W^{\text{cs}}$ is independent of any two different choices in $\beta$. Also, $W^{\text{cs}}$ is the graph of a $C^1$ function $\phi_u : \mathbb{E}^{cs} \to \mathbb{E}^u$*

$$W^{\text{cs}} = \text{graph}(\phi_u),$$

*and the tangent space of $W^{\text{cs}}$ at the fixed point is the center-stable eigenspace*

$$\mathbb{T}_{\bar{q}} W^{\text{cs}} \cong \mathbb{E}^{cs}.$$

*Furthermore, if $f \in C^k(\mathbb{R}^d)$, $1 \le k < \infty$, then $\phi_u \in C^k(\mathbb{E}^{cs}, \mathbb{E}^u)$, and if $f \in C^{k,1}(\mathbb{R}^d)$, then $\phi_u \in C^{k,1}(\mathbb{E}^{cs}, \mathbb{E}^u)$.*

The proof is an application of the Uniform Contraction Principle. The main idea is to construct the center-stable manifold function $\phi_u$ as part of a fixed point of a uniform contraction map. We will break it up into a few lemmas.

Before doing so, we recall a few important properties about $f$. We first translate $\bar{q}$ to the origin and choose a coordinate system $(x, y)$ for the splitting $\mathbb{R}^d \cong \mathbb{E}^{cs} \times \mathbb{E}^u$ for which $Df(\bar{q}) \cong \text{diag}(A_{cs}, A_u)$. By the Variation of Parameters Formula Theorem, a sufficiently small $\|f - Df(\bar{q})\|_1$ implies that the map $(\bar{x}, \bar{y}) = f(x, y)$ is equivalent to

$$\begin{cases} \bar{x} = A_{cs} x + h_{cs}(x, y) \\ y = A_u^{-1} \bar{y} + h_u(\bar{x}, \bar{y}), \end{cases} \tag{1}$$

and for any orbit, $p_n = (x_n, y_n) = f(x_{n-1}, y_{n-1}),\ n \geq 0$,

$$\begin{cases} x_n = A_{cs}^n x_0 + \sum_{i=1}^n A_{cs}^{n-i} h_{cs}(p_{i-1}) \\ y_n = A_u^{n-m} y_m + \sum_{i=n+1}^m A_u^{n+1-i} h_u(p_i). \end{cases} \tag{2}$$

Also, by the VPF theorem, functions $h_{cs}, h_u$ are all $C^1$ satisfying

$$h_{cs}(0) = 0,\ Dh_{cs}(0) = 0,\ h_u(0) = 0,\ Dh_u(0) = 0 \tag{3}$$

and they are globally Lipschitz and the Lipschitz constant can be taken to be

$$L = \|Dh\|_0 \to 0 \quad \text{as} \quad \|f - Df(\bar{q})\|_1 \to 0. \tag{4}$$

We will repeatedly use the formula below and and its differentiations in $r$

$$a + ar + ar^2 + \cdots + ar^{n-1} = \tfrac{a(1-r^n)}{1-r}, \text{ for } r \neq 1.$$

**Lemma 1.** *For the parameter $\beta$ from the definition of $W^{\mathrm{cs}}$, let*

$$S_\beta := \{\gamma = \{p_n\}_{n=0}^\infty : p_n \in \mathbb{R}^d, \sup\{\beta^{-n}\|p_n\| : n \geq 0\} < \infty\} \tag{5}$$

*with norm*

$$\|\gamma\|_\beta = \sup\{\beta^{-n}\|p_n\| : n \geq 0\}.$$

*For any $\gamma = \{p_n = (x_n, y_0)\} \in S_\beta$, let $\overline{\gamma} = T(\gamma)$ be defined by equations*

$$\begin{cases} \bar{x}_n = A_{cs}^n x_0 + \sum_{i=1}^n A_{cs}^{n-i} h_{cs}(p_{i-1}) \\ \bar{y}_n = \sum_{i=n+1}^\infty A_u^{n+1-i} h_u(p_i). \end{cases} \tag{6}$$

*Then $\overline{\gamma} \in S_\beta$ with*

$$\|\bar{\gamma}\|_\beta \leq \|x_0\| + \tfrac{L\|\gamma\|_\beta}{\beta - \varsigma} + \tfrac{L\beta\|\gamma\|_\beta}{1 - \alpha\beta} \tag{7}$$

*where parameters $\alpha, \varsigma$ and $\beta$ satisfy*

$$1/\min\{|\sigma^u|\} < \alpha < 1 < \varsigma < \beta < 1/\alpha < \min\{|\sigma^u|\}. \tag{8}$$

*More importantly, $p = (x_0, y_0) \in W^{\mathrm{cs}}$ if and only if the orbit $\gamma_p = \{f^n(p)\}_{n=0}^\infty$ is a fixed point of $T$ and*

$$p = (x_0, y_0) = \left(x_0, \textstyle\sum_{i=1}^\infty A_u^{1-i} h_u(p_i)\right). \tag{9}$$

*Proof.* An adapted norm will be chosen throughout, but for this lemma we only need it to satisfy the relations below

$$\|A_u^{-1}\| < \alpha < 1,\ \|A_{cs}\| < \varsigma < \beta < 1/\alpha < \|A_u\|. \tag{10}$$

We now show $\bar{\gamma} \in S_\beta$. Specifically, because $\|h_{cs}(p)\| = \|h_{cs}(p) - h_{cs}(0)\| \leq L\|p\|$ and $1 < \varsigma < \beta$, we have for $\bar{x}_n$

$$\begin{aligned} \|\bar{x}_n\| &\leq \|A_{cs}^n\|\|x_0\| + \textstyle\sum_{i=1}^n \|A_{cs}^{n-i} h_{cs}(p_{i-1})\| \\ &\leq \varsigma^n\|x_0\| + \textstyle\sum_{i=1}^n \varsigma^{n-i} L\beta^{i-1}\|\gamma\|_\beta \\ &= \varsigma^n\|x_0\| + L\|\gamma\|_\beta \tfrac{\beta^n - \varsigma^n}{\beta - \varsigma} \leq (\|x_0\| + \tfrac{L\|\gamma\|_\beta}{\beta - \varsigma})\beta^n. \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} \|\bar{y}_n\| &\le \textstyle\sum_{i=n+1}^{\infty} \|A_u^{n+1-i} h_u(p_i)\| \le \sum_{i=n+1}^{\infty} \alpha^{i-n-1} L \beta^i \|\gamma\|_\beta \\ &= \alpha^{-n-1} L \|\gamma\|_\beta \tfrac{(\alpha\beta)^{n+1}}{1-\alpha\beta} = \tfrac{L\beta\|\gamma\|_\beta}{1-\alpha\beta}\beta^n. \end{aligned} \tag{12}$$

Hence, the bound estimate (7) holds, implying $T : S_\beta \to S_\beta$.

Next, for any $p := p_0 = (x_0, y_0) \in W^{\mathrm{cs}}$, by definition $\gamma = \{p_n = f^n(p_0)\} \in S_\beta$, so $\|p_n\| \le \|\gamma\|_\beta \beta^n$ for $n \ge 0$. Because for $m \ge n$, $\|A_u^{n-m}\| \le \alpha^{m-n}$, and $\alpha\beta < 1$, the first term of the $y_n$-equation of the VPF (2) goes to 0 as $m \to \infty$. Since $h_u$ is bounded, the partial sum of the $y_n$-equation converges as well for $m \to \infty$. So every orbit from $W^{\mathrm{cs}}$ satisfies

$$\begin{cases} x_n = A_{cs}^n x_0 + \sum_{i=1}^{n} A_{cs}^{n-i} h_{cs}(p_{i-1}) \\ y_n = \sum_{i=n+1}^{\infty} A_u^{n+1-i} h_u(p_i), \end{cases} \tag{13}$$

showing $\gamma$ is a fixed point of $T$.

Conversely, if a sequence $\gamma = \{p_n = (x_n, y_n)\} \in S_\beta$ is a fixed point of $T$, satisfying (13), then it is straightforward to verify

$$x_{n+1} = A_{cs} x_n + h_{cs}(x_n, y_n) \text{ and } y_n = A_u^{-1} y_{n+1} + h_u(x_{n+1}, y_{n+1})$$

hold for all $n \ge 0$. By (1) the sequence is an orbit of $f$. Therefore, $\gamma \in W^{\mathrm{cs}}$ by definition. □

**Lemma 2.** *There is a Lipschitz continuous function $\phi_u \in C^{0,1}(\mathbb{E}^{cs}, \mathbb{E}^u)$ so that*

$$W^{\mathrm{cs}} = \mathrm{graph}(\phi_u). \tag{14}$$

*Proof.* By Lemma 1, we know that $p \in W^{\mathrm{cs}}$ if and only if $p$ is the initial point of a sequence $\gamma \in S_\beta$ which is a fixed point of the map $T$ defined by (6) and (9) holds. To show the existence of such a fixed point, we will consider $T$ as a parameterized map by $x_0 \in \mathbb{E}^{cs}$ and show that $T(\cdot, x_0) : S_\beta \to S_\beta,\ x_0 \in \mathbb{E}^{cs}$, is a uniform contraction. Specifically, let $\gamma, \gamma'$ and $\bar{\gamma} = T(\gamma, x_0), \bar{\gamma}' = T(\gamma', x_0)$. We have

$$\begin{aligned} \|\bar{x}_n - \bar{x}'_n\| &\le \textstyle\sum_{i=1}^{n} \|A_{cs}^{n-i}[h_{cs}(p_{i-1}) - h_{cs}(p'_{i-1})]\| \\ &\le \textstyle\sum_{i=1}^{n} \varsigma^{n-i} L \|p_{i-1} - p'_{i-1}\| \\ &\le \textstyle\sum_{i=1}^{n} \varsigma^{n-i} L \beta^{i-1} \|\gamma - \gamma'\|_\beta \\ &\le \tfrac{L}{\beta-\varsigma}\beta^n \|\gamma - \gamma'\|_\beta \end{aligned} \tag{15}$$

and

$$\begin{aligned} \|\bar{y}_n - \bar{y}'_n\| &\le \textstyle\sum_{i=n+1}^{\infty} \|A_u^{n+1-i}[h_u(p_i) - h_u(p'_i)]\| \\ &\le \textstyle\sum_{i=n+1}^{\infty} \alpha^{i-n-1} L \|p_i - p'_i\| \\ &\le \textstyle\sum_{i=n+1}^{\infty} \alpha^{i-n-1} \beta^i \|\gamma - \gamma'\|_\beta \\ &\le \tfrac{L\beta}{1-\alpha\beta}\beta^n \|\gamma - \gamma'\|_\beta\ . \end{aligned} \tag{16}$$

Hence,

$$\|T(\gamma, x_0) - T(\gamma', x_0)\|_\beta \le (\tfrac{L}{\beta-\varsigma} + \tfrac{L\beta}{1-\alpha\beta})\|\gamma - \gamma'\|_\beta\ ,$$

showing $T(\cdot, x_0)$ is a uniform contraction provided

$$\theta := \theta(\beta) = \tfrac{L}{\beta-\varsigma} + \tfrac{L\beta}{1-\alpha\beta} < 1 \tag{17}$$

which is true for small $\|f - Df(\bar{q})\|_1$ by (4). Denote the unique fixed point of $T(\cdot, x_0)$ by

$$\gamma^*(x_0) = \{p_n(x_0)\}_{n=0}^{\infty}, \;\; p_n(x_0) = (x_n(x_0), y_n(x_0)), \; n \geq 0. \tag{18}$$

Define

$$\phi_u(x_0) := y_0(x_0) = \textstyle\sum_{i=1}^{\infty} A_u^{1-i} h_u(p_i(x_0)), \tag{19}$$

the $y$-coordinate of the initial point of the fixed point $\gamma^*(x_0)$. By Lemma 1(9), we have $p \in W^{\mathrm{cs}}$ iff $p = (x_0, y_0) = (x_0, \phi_u(x_0))$, i.e., the identity (14).

Next, since $T : S_\beta \times \mathbb{E}^{cs} \to S_\beta$ is Lipschitz continuous in $x_0$ with

$$\|T(\gamma, x_0) - T(\gamma, x_0{}')\|_\beta \leq \|x_0 - x_0{}'\|$$

because $\|A_{cs}^n\| < \beta^n$, we have by the Uniform Contraction Principle I that $\gamma^*(x_0)$ is Lipschitz continuous with

$$\|\gamma^*(x_0) - \gamma^*(x_0{}')\|_\beta \leq \tfrac{1}{1-\theta}\|x_0 - x_0{}'\| \tag{20}$$

which in turn implies $\phi_u$ is Lipschitz continuous with

$$\|\phi_u(x_0) - \phi_u(x_0{}')\| \leq \|\gamma^*(x_0) - \gamma^*(x_0{}')\|_\beta \leq \tfrac{1}{1-\theta}\|x_0 - x_0{}'\| \,,$$

completing the proof of the lemma. □

**Lemma 3.** *$\phi_u \in C^1(\mathbb{E}^{cs}, \mathbb{E}^u)$ and $\mathbb{T}_{\bar{q}} W^{\mathrm{cs}} = \mathbb{E}^{cs}$.*

*Proof.* The main argument is to show that the Uniform Contraction Principle II applies to $T$ for $k = 1$. Two conditions are needed to verify: (1) $T \in C^1(S_\beta \times \mathbb{E}^{cs}, S_\beta)$; and (2) $\|D_\gamma T(\gamma, x_0)\|$ is uniformly bounded by a constant smaller than 1.

To verify the conditions, let $\gamma = \{p_n\}, v = \{v_n\} \in S_\beta$, and formally differentiate (6). Then $D_\gamma T(\gamma, x_0)v$ needs to be as below in components:

$$\begin{cases} [D_\gamma T(\gamma, x_0)v]_{n,\, cs} = \sum_{i=1}^{n} A_{cs}^{n-i} Dh_{cs}(p_{i-1})v_{i-1} \\ [D_\gamma T(\gamma, x_0)v]_{n,\, u} = \sum_{i=n+1}^{\infty} A_u^{n+1-i} Dh_u(p_i)v_i. \end{cases} \tag{21}$$

By exactly the same estimates as for (15, 16) we have

$$\|[D_\gamma T(\gamma, x_0)v]_{n,\, cs}\| \leq \tfrac{L}{\beta-\varsigma}\beta^n\|v\|_\beta$$

and

$$\|[D_\gamma T(\gamma, x_0)v]_{n,\, u}\| \leq \tfrac{L\beta}{1-\alpha\beta}\beta^n\|v\|_\beta \,.$$

These estimates imply three things. One, because of the uniform convergence of the second equation, the derivative $D_\gamma T(\gamma, x_0)$ is well-defined. Two, the derivative is in fact in $L(S_\beta, S_\beta)$ as required. Three, the derivative's $\beta$-norm

$$\|D_\gamma T(\gamma, x_0)\|_\beta \le \theta(\beta) < 1$$

is bounded by the same uniform contraction constant $\theta(\beta)$. About its derivative in $x_0$, we have

$$[D_{x_0} T(\gamma, x_0)]_{n,\ cs} = A_{cs}^n, \text{ and } [D_{x_0} T(\gamma, x_0)]_{n,\ u} = 0.$$

Obviously, $D_{x_0} T(\gamma, x_0) \in L(\mathbb{E}^{cs}, S_\beta)$ since $\|A_{cs}^n\| < \beta^n$. This shows the Uniform Contraction Principle II indeed applies for $T$ with the case of $k = 1$. Thus, we can conclude that for the fixed point, $\gamma^*(\cdot) \in C^1(\mathbb{E}^{cs}, S_\beta)$, and $\phi_u \in C^1(\mathbb{E}^{cs}, \mathbb{E}^u)$ follows.

Furthermore, since the fixed point $\bar{q} \sim 0$ is obviously on the manifold, we have $\gamma^*(0) = \{0\}_{n\ge 0}$, the zero sequence. Hence, $\phi_u(0) = 0$. In addition, for the derivative of $\phi_u$, we have from (19)

$$D\phi_u(x_0) = \textstyle\sum_{i=1}^\infty A_u^{1-i} Dh_u(p_i(x_0)) Dp_i(x_0).$$

Because $Dh_u(0) = 0$, and $p_i(0) = 0$ for all $i \ge 0$, we have

$$D\phi_u(0) = 0,$$

showing that the tangent space of $W^{\mathrm{cs}}$ at $\bar{q} \sim 0$ is the center-stable eigenspace $\mathbb{R}^{d_{cs}} \cong \mathbb{E}^{cs}$. This proves the theorem for $k = 1$. □

**Lemma 4.** *The definition of $W^{\mathrm{cs}}$ is independent of any two choices in $\beta$. More specifically, let $\gamma^*(x_0)$ be the fixed point of the map $T(\cdot, x_0)$ from Lemma 1, then for any $1 < \beta' < \beta$, a sufficiently small $\|f - Df(\bar{q})\|_1$ implies $\gamma^*(\cdot) \in C^1(\mathbb{E}^{cs}, S_{\beta'})$ and $\gamma^*(\cdot) \in C^1(\mathbb{E}^{cs}, S_\beta)$.*

*Proof.* Let $\beta'$ and $\beta$ be two different constants satisfying the definition of $W^{\mathrm{cs}}$. Assume without loss of generality that $1 < \beta' < \beta < \min\{|\sigma^u|\}$. On one hand, it is automatically true by definition that

$$W^{\mathrm{cs}}_{\beta'} \subseteq W^{\mathrm{cs}}_\beta$$

because $S_{\beta'} \subset S_\alpha$ for $\beta' < \beta$.

On the other hand, we can re-adjust the adapted norm if necessary so that

$$\|A_{cs}\| < \varsigma < \beta' < \beta < 1/\alpha < \min\{|\sigma^u|,\ \|A_u^{-1}\| < \alpha < 1.$$

Also, by making $\|f - Df(\bar{q})\|_1$ smaller if necessary, we can assume

$$\theta(\beta'),\ \theta(\beta) < 1.$$

Thus, the same estimates (11, 12) imply that the uniform contraction map $T(\cdot, x_0)$ defined in $S_\beta$ maps the subset $S_{\beta'}$ into itself. Therefore, the fixed point function $\gamma^*(\cdot)$ for parameter $\beta$ must reside in $S_{\beta'}$, and therefore the reverse inclusion $W^{\mathrm{cs}}_\beta \subseteq W^{\mathrm{cs}}_{\beta'}$ follows, implying

$$W^{\mathrm{cs}}_{\beta'} = W^{\mathrm{cs}}_{\beta} \ ,$$

i.e., the independence of $W^{\mathrm{cs}}$ on $\beta$. The proof of Lemma 3 also shows the same fixed point function $\gamma^*(\cdot)$ is in both $C^1(\mathbb{E}^{cs}, S_{\beta'})$ and $C^1(\mathbb{E}^{cs}, S_\beta)$. □

**Lemma 5.** *If $f \in C^k(\mathbb{R}^d)$, then $\phi_u \in C^k(\mathbb{E}^{cs}, \mathbb{E}^u)$. If $f \in C^{k,1}$, then $\phi_u \in C^{k,1}$.*

*Proof.* The $k = 1$ case is proved in Lemma 3. For $k \geq 2$, we note that the Uniform Contraction Principle II cannot apply directly as the proof of Lemma 3 did for $k = 1$. This is because we cannot prove $T \in C^k(S_\beta \times \mathbb{E}^{cs}, S_\beta)$. An indirect approach is needed.

We begin by choosing a constant $\mu$ as below

$$1 < \mu = \beta^{1/(k+1)} < \beta \tag{22}$$

and assume

$$\|A_{cs}\| < \varsigma < \mu < \beta < 1/\alpha, \ \|A_u^{-1}\| < \alpha < 1, \tag{23}$$

by re-adjusting the adapted norm if necessary. By Lemma 4, we have for small $\|f - Df(\bar{q})\|_1$ and $\beta' = \mu$ the following

$$\gamma^*(\cdot) \in C^1(\mathbb{E}^{cs}, S_\mu) \ \text{ and } \ T \in C^1(S_\mu \times \mathbb{E}^{cs}, S_\mu). \tag{24}$$

We want to prove first instead the following claim

$$T \in C^k(S_\mu \times \mathbb{E}^{cs}, S_\beta). \tag{25}$$

We note first that

$$[D_{x_0} T(\gamma, x_0)]_{n,\, cs} = A^n_{cs}, \ \text{ and } \ [D_{x_0} T(\gamma, x_0)]_{n,\, u} = 0.$$

This implies any mixed derivative in $\gamma$ and $x_0$ are the zero operators, hence well-defined and exists. So, we only need to show $T$ is $C^k(S_\mu \times \mathbb{E}^{cs}, S_\beta)$ separately in $\gamma$ and $x_0$. For the latter, the identity above shows

$$\|[D_{x_0} T(\gamma, x_0)]_n\| \leq \|A^n_{cs}\| < \mu^n < \beta^n$$

and $\|D_{x_0} T(\gamma, x_0]\|_\beta \leq 1$ follows. Also, $D^j_{x_0} T(\gamma, x_0) = 0$, for $2 \leq j \leq k$. Hence, $T(\gamma, \cdot) \in C^k(\mathbb{E}^{cs}, S_\beta)$.

Now we show $T(\cdot, x_0) \in C^k(S_\mu, S_\beta)$, i.e., $D^j_\gamma T(\gamma, x_0)$ is a bounded $j$-linear form from $\otimes^j S_\mu$ to $S_\beta$ for any $1 \leq j \leq k$. The case of $j = 1$ is true by (24) because $T(\cdot, x_0) \in C^1(S_\mu, S_\mu) \subset C^1(S_\mu, S_\beta)$ since $S_\mu \subset S_\beta$ for $1 < \mu < \beta$.

For any $2 \leq j \leq k$, $[D^j_\gamma T(\gamma, x_0)]$ should be a bounded $j$-linear form from $S_\mu$ to $S_\beta$. To this end, let $v = v^1 \otimes v^2 \otimes \cdots \otimes v^j$ with each $v^\ell \in S_\mu$. Formally differentiate (6) to get

$$\left\{ \begin{array}{l} [D^j_\gamma T(\gamma, x_0) v]_{n,\, cs} = \sum_{i=1}^n A^{n-i}_{cs} D^j h_{cs}(p_{i-1}) v_{i-1} \\ {[D^j_\gamma T(\gamma, x_0) v]}_{n,\, u} = \sum_{i=n+1}^\infty A^{n+1-i}_u D^j h_u(p_i) v_i, \end{array} \right. \tag{26}$$

where

$$v_i = v_i^1 \otimes v_i^2 \otimes \cdots \otimes v_i^j, \quad v_i^\ell \in \mathbb{R}^d.$$

Similar to the estimate of (15), we have

$$\begin{aligned} \|[D_\gamma^j T(\gamma, x_0)v]_{n,\, cs}\| &\leq \textstyle\sum_{i=1}^n \|A_{cs}^{n-i}\| \|[D^j h_{cs}(p_{i-1})]v_{i-1}\| \\ &\leq \textstyle\sum_{i=1}^n \varsigma^{n-i} \|h_{cs}\|_j \Pi_{\ell=1}^j \|v_{i-1}^\ell\| \\ &\leq \|h_{cs}\|_k \textstyle\sum_{i=1}^n \varsigma^{n-i} \mu^{j(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\mu \\ &\leq \|h_{cs}\|_k \textstyle\sum_{i=1}^n \varsigma^{n-i} \mu^{k(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\mu \\ &\leq \|h_{cs}\|_k \textstyle\sum_{i=1}^n \varsigma^{n-i} \beta^{i-1} \Pi_{\ell=1}^j \|v^\ell\|_\mu \\ &\leq \tfrac{\|h_{cs}\|_k}{\beta-\varsigma} \beta^n \Pi_{\ell=1}^j \|v^\ell\|_\mu \end{aligned} \tag{27}$$

where $\|A_{cs}\| < \varsigma < \mu < \beta$ and $\mu^k < \beta$ by (22, 23). Similar to the estimate of (16) we have

$$\begin{aligned} \|[D_\gamma^j T(\gamma, x_0)v]_{n,\, u}\| &\leq \textstyle\sum_{i=n+1}^\infty \|A_u^{n+1-i}\| \|[D^j h_u(p_i)]v_i\| \\ &\leq \textstyle\sum_{i=n+1}^\infty \alpha^{i-n-1} \|h_u\|_j \mu^{ji} \Pi_{\ell=1}^j \|v^\ell\|_\mu \\ &\leq \|h_u\|_k \alpha^{-n-1} \textstyle\sum_{i=n+1}^\infty (\alpha\mu^j)^i \Pi_{\ell=1}^j \|v^\ell\|_\mu \\ &\leq \|h_u\|_k \alpha^{-n-1} \textstyle\sum_{i=n+1}^\infty (\alpha\mu^k)^i \Pi_{\ell=1}^j \|v^\ell\|_\mu \\ &\leq \|h_u\|_k \alpha^{-n-1} \tfrac{(\alpha\beta)^{n+1}}{1-\alpha\beta} \Pi_{\ell=1}^j \|v^\ell\|_\mu \\ &\leq \tfrac{\|h_u\|_k \beta}{1-\alpha\beta} \beta^n \Pi_{\ell=1}^j \|v^\ell\|_\mu. \end{aligned} \tag{28}$$

Combine these two estimates to obtain

$$\|[D_\gamma^j T(\gamma, x_0)]\|_\beta \leq \|(h_{cs}, h_u)\|_k \max\{\tfrac{1}{\beta-\varsigma}, \tfrac{\beta}{1-\alpha\beta}\}.$$

The convergence of the infinite series also shows the derivatives are well-defined. This completes the proof that $T \in C^k(S_\mu \times \mathbb{E}^{cs}, S_\beta)$.

We are now ready to show $\gamma^*(\cdot) \in C^k(\mathbb{E}^{cs}, S_\beta)$. By the Uniform Contraction Principle II for $T \in C^1(S_\mu \times \mathbb{E}^{cs}, S_\mu)$, the fixed point $\gamma^*(\cdot)$ is in $C^1(\mathbb{E}^{cs}, S_\mu)$ and its derivative is given by

$$D\gamma^*(\cdot) = \textstyle\sum_{n=0}^\infty [D_\gamma T(\gamma^*(\cdot), \cdot)]^n D_{x_0} T(\gamma^*(\cdot), \cdot).$$

Since $\gamma^*(\cdot) \in C^1(\mathbb{E}^{cs}, S_\mu)$, $T \in C^1(S_\mu \times \mathbb{E}^{cs}, S_\mu) \subset C^1(S_\mu \times \mathbb{E}^{cs}, S_\beta)$, and $T \in C^k(S_\mu \times \mathbb{E}^{cs}, S_\beta), k \geq 2$, here is the key to notice that the composition $D_\gamma T(\gamma^*(\cdot), \cdot)$ is $C^1(\mathbb{E}^{cs}, S_\beta)$. This implies that the infinite series on the right is in $C^1(\mathbb{E}^{cs}, S_\beta)$, and therefore, $D\gamma^*(\cdot) \in C^1(\mathbb{E}^{cs}, S_\beta)$, and $\gamma^*(\cdot) \in C^2(\mathbb{E}^{cs}, S_\beta)$ follows. Apply this argument recursively to obtain $\gamma^*(\cdot) \in C^3(\mathbb{E}^{cs}, S_\beta)$, and so on until we reach $\gamma^*(\cdot) \in C^k(\mathbb{E}^{cs}, S_\beta)$. As a component of the initial point of $\gamma^*$, $\phi_u$ is in $C^k(\mathbb{E}^{cs}, \mathbb{E}^u)$ as well.

For the case of $f \in C^{k,1}$, the argument above can be used to show first $T \in C^{k,1}(S_\mu \times \mathbb{E}^{cs}, S_\beta)$, using $\mu^{k+1} = \beta$, and then $\gamma^* \in C^{k,1}(\mathbb{E}^{cs}, S_\beta)$, which in turn implies $\phi_u$ is $C^{k,1}$. This completes the proof. □

The lemmas above complete the proof for Theorem 1. For future reference, we state the following result from the proofs above.

**Proposition 1.** *For any* $1 < \mu < \beta < \min\{|\sigma^u|\}$ *and small* $\|f - Df(\bar{q})\|_1$*, the orbit* $\gamma_p = \{f^n(p)\}_{n=0}^{\infty}$ *of any point* $p = (x_0, y_0) \in W^{\text{cs}}$ *can be expressed as a function* $\gamma_p = \gamma^*(x_0)$ *for* $x_0 \in \mathbb{E}^{cs}$ *so that* $\gamma^* \in C^k(\mathbb{E}^{cs}, S_\mu)$ *and* $\gamma^* \in C^k(\mathbb{E}^{cs}, S_\beta)$ *if* $f \in C^k(\mathbb{R}^d), 1 \leq k < \infty$*, or* $\gamma^* \in C^{k,1}(\mathbb{E}^{cs}, S_\mu)$ *and* $\gamma^* \in C^{k,1}(\mathbb{E}^{cs}, S_\beta)$ *if* $f \in C^{k,1}(\mathbb{R}^d)$.

**Definition 2.** *Let* $\bar{q}$ *be a nonhyperbolic fixed point of a diffeomorphism* $f$ *in* $\mathbb{R}^d$ *and* $\alpha$ *be any constant satisfying*

$$\max\{|\sigma^s|\} < \alpha < 1.$$

*The center-unstable manifold of the fixed point* $\bar{q}$ *is*

$$W^{\text{cu}} = \{p : \{\alpha^n[f^{-n}(p) - \bar{q}]\}_{n=0}^{\infty} \textit{ is a bounded sequence}\}.$$

By applying Theorem 1 to $f^{-1}$ we obtain the following result.

**Theorem 2** (Center-Unstable Manifold Theorem)**.** *Let* $\bar{q}$ *be a nonhyperbolic fixed point of a diffeomorphism* $f$ *in* $\mathbb{R}^d$ *with splitting* $\mathbb{R}^d \cong \mathbb{E}^s \times \mathbb{E}^{cu}$*. Then a sufficiently small* $\|f - Df(\bar{q})\|_1$ *implies* $W^{\text{cu}}$ *is independent of any two different choices in* $\alpha$*. Also,* $W^{\text{cu}}$ *is the graph of a* $C^1$ *function* $\phi_s : \mathbb{E}^{cu} \to \mathbb{E}^s$

$$W^{\text{cu}} = \text{graph}(\phi_s),$$

*and the tangent space of* $W^{\text{cu}}$ *at the fixed point is the center-unstable eigenspace*

$$\mathbb{T}_{\bar{q}} W^{\text{cu}} \cong \mathbb{E}^{cu}.$$

*Furthermore, if* $f \in C^k(\mathbb{R}^d)$*,* $1 \leq k < \infty$*, then* $\phi_s \in C^k(\mathbb{E}^{cu}, \mathbb{E}^s)$*, and if* $f \in C^{k,1}(\mathbb{R}^d)$*, then* $\phi_s \in C^{k,1}(\mathbb{E}^{cu}, \mathbb{E}^s)$.

**Theorem 3** (Local Center-stable and Local Center-unstable Manifold Theorem)**.** *Let* $\bar{q}$ *be a nonhyperbolic fixed point of a diffeomorphism* $f$ *in* $\mathbb{R}^d$ *and let* $\mathbb{E}^{cs}$*,* $\mathbb{E}^{cu}$*,* $\mathbb{E}^s$*,* $\mathbb{E}^u$ *be the center-stable, center-unstable, stable, unstable eigenspace, respectively, at* $\bar{q}$ *for the linearization* $Df(\bar{q})$*. Then there is a small neighborhood* $N_r(\bar{q})$ *and two differentiable functions* $\phi_u : N_r(\bar{q}) \cap \mathbb{E}^{cs} \to \mathbb{E}^u$*,* $\phi_s : N_r(\bar{q}) \cap \mathbb{E}^{cu} \to \mathbb{E}^s$*, so that the local center-stable and local center-unstable manifolds*

$$W^{\text{cs}}_{\text{loc}}(\bar{q}) := \text{graph}(\phi_u), \quad W^{\text{cu}}_{\text{loc}}(\bar{q}) := \text{graph}(\phi_s)$$

*satisfy the following properties*

*(i)* $W^{\text{cs}}_{\text{loc}}$ *contains all bounded forward orbits in* $N_r$*.*

*(ii)* $W^{\text{cu}}_{\text{loc}}$ *contains all bounded backward orbits in* $N_r$*.*

*(iii) They are locally invariant, i.e.,* $f(W^i_{\text{loc}}) \cap N_r \subseteq W^i_{\text{loc}}$*,* $f^{-1}(W^i_{\text{loc}}) \cap N_r \subseteq W^i_{\text{loc}}$ *,* $i = \text{cs, cu}$

*(iv)* $\mathbb{T}_{\bar{q}} W^{\text{cs}}_{\text{loc}} \cong \mathbb{E}^{cs}$*,* $\mathbb{T}_{\bar{q}} W^{\text{cu}}_{\text{loc}} \cong \mathbb{E}^{cu}$*.*

*Moreover, if $f$ is $C^k$, $1 \leq k < \infty$, then both $\phi_u$ and $\phi_s$ are $C^k$, and if $f$ is $C^{k,1}$, then both $\phi_u$ and $\phi_s$ are $C^{k,1}$.*

*Proof.* Modify the map $f$ by a $C^\infty$ cut-off function $\rho_r(p - \bar{q})$ to $f \to f(p) = Df(\bar{q})p + \rho_r(p - \bar{q})(f(p) - Df(\bar{q})p)$. Then for sufficiently small $r$, Theorems 1 and 2 can be applied to the modified map to obtain the maps $\phi_u, \phi_s$. Restrict both to the neighborhood $N_r(\bar{q})$, then the results follow from the theorems. □

By applying the theorem above we obtain

**Theorem 4** (Local Center Manifold Theorem)**.** *Let $\bar{q}$ be a nonsingular fixed point of a continuously differentiable map $f$ in $\mathbb{R}^d$ and let $\mathbb{E}^s$, $\mathbb{E}^c$, $\mathbb{E}^u$ be the stable, center, unstable eigenspace, respectively, at $\bar{q}$ for the linearization $Df(\bar{q})$. Then there is a small neighborhood $N_r(\bar{q})$ and a differentiable function $\phi_{su} : N_r(\bar{q}) \cap \mathbb{E}^c \to \mathbb{E}^s \times \mathbb{E}^u$, so that the local center manifold*

$$W^{\mathrm{c}}_{\mathrm{loc}}(\bar{q}) := \mathrm{graph}(\phi_{su})$$

*satisfies the following properties*

*(i)* *$W^{\mathrm{c}}_{\mathrm{loc}}$ contains all orbits bounded in both forward and backward directions in $N_r$.*

*(ii)* *Every point not from $W^{\mathrm{c}}_{\mathrm{loc}}$ escapes $N_r(\bar{q})$ in either forward or backward iteration.*

*(iii)* *It is locally invariant, $f(W^{c}_{\mathrm{loc}}) \cap N_r \subseteq W^{c}_{\mathrm{loc}}$, $f^{-1}(W^{c}_{\mathrm{loc}}) \cap N_r \subseteq W^{c}_{\mathrm{loc}}$.*

*(iv)* *$\mathbb{T}_{\bar{q}} W^{\mathrm{c}}_{\mathrm{loc}} \cong \mathbb{E}^c$.*

*Moreover, if $f$ is $C^k$, $1 \leq k < \infty$, then $\phi_{su}$ is $C^k$, and if $f$ is $C^{k,1}$, then $\phi_{su}$ is $C^{k,1}$.*

*Proof.* Let $W^{\mathrm{cs}}_{\mathrm{loc}} = \mathrm{graph}(\phi_u)$ and $W^{\mathrm{cu}}_{\mathrm{loc}} = \mathrm{graph}(\phi_s)$ be a local center-stable manifold and a local center-unstable manifold, respectively, by the previous theorem. Define

$$W^{\mathrm{c}}_{\mathrm{loc}} = W^{\mathrm{cs}}_{\mathrm{loc}} \cap W^{\mathrm{cu}}_{\mathrm{loc}}.$$

Then property (i) through (iii) follow immediately. To show the existence of $\phi_{su}$ and (iv), let $p = (x, y, z)$ be a coordinate system for the splitting $\mathbb{R}^d = \mathbb{E}^s \times \mathbb{E}^c \times \mathbb{E}^u$. Then a point $(x, y, z) \in W^{\mathrm{c}}_{\mathrm{loc}}$ iff it satisfies the equations below

$$\begin{cases} x = \phi_s(y, z) \\ z = \phi_u(x, y) \end{cases} \tag{29}$$

which in turn is equivalent to $F(x, y, z) = (F_1, F_2)(x, y, z) = 0$ with

$$F_1(x, y, z) = x - \phi_s(y, z), \ \text{ and } \ F_2(x, y, z) = z - \phi_u(x, y).$$

Obviously, the fixed point, $\bar{q} \sim (0, 0, 0)$, is a solution, $F(0, 0, 0) = 0$. Also,

$$D_{(x,z)} F(0, 0, 0) = I$$

the identity matrix in $\mathbb{R}^{d_s+d_u} \cong \mathbb{E}^s \times \mathbb{E}^u$, because $D\phi_u(0,0) = 0$ and $D\phi_s(0,0) = 0$. Therefore, by the Implicit Function Theorem, equation (29), i.e. $F(x,y,z) = 0$, can be solved locally as a function $\phi_{su} : N_r \cap \mathbb{E}^c \to \mathbb{E}^s \times \mathbb{E}^u$, making $r$ smaller if necessary, so that $(x,z) = \phi_{su}(y)$ and

$$W^{\mathrm{c}}_{\mathrm{loc}} = \mathrm{graph}(\phi_{su})$$

follows. It can be directly checked that $\phi_{su}(0) = (0,0)$ and

$$D\phi_{su}(0) = 0$$

by IFT since $D_yF(0,0,0) = 0$, showing property (iv). Last, that $f$ is $C^k$, or $C^{k,1}$, $1 \le k < \infty$, implies $\phi_u, \phi_s$ are $C^k$, or $C^{k,1}$, which in turn by IFT implies $\phi_{su}$ is $C^k$, or $C^{k,1}$. This completes the proof. □

The conclusion is all interesting dynamics near a nonhyperbolic fixed point of a diffeomorphism takes place on a center manifold.

Local center manifolds are not unique in general (see Fig.1), but the center manifold dynamics is in the sense that the dynamics on any two local center manifolds are smoothly conjugate. Specifically, we have the following theorem.

**Theorem 5** (Uniqueness of Center Manifold Dynamics for Flow [1])**.** *Let $\bar{q} = 0$ be a nonhyperbolic equilibrium point of the differential equation*

$$\dot{x} = Ax + h(x)$$

*where $x \in \mathbb{R}^d$, $h(0) = 0$, $Dh(0) = 0$, and $h$ is $C^{k+1,1}, k \ge 0$. Let $f$ be the time-1 map of the solution, $f(x) = \varphi(1,x)$ where $\varphi(t,x_0)$ is the solution of the equation with initial condition $\varphi(0,x_0) = x_0$. Let $W^c_{\mathrm{loc},1}$, $W^c_{\mathrm{loc},2}$ be two local center manifolds of $\bar{q}$ for $f$. Then there is an open neighborhood $V$ of $\bar{q}$ and a $C^k$ invertible map $\kappa : W^c_{\mathrm{loc},1} \cap V \to W^c_{\mathrm{loc},2} \cap V$ so that*

$$f \circ \kappa(p) = \kappa \circ f(p)$$

*for all $p \in W^c_{\mathrm{loc},1} \cap V$ so long as $f(p) \in W^c_{\mathrm{loc},1} \cap V$.*

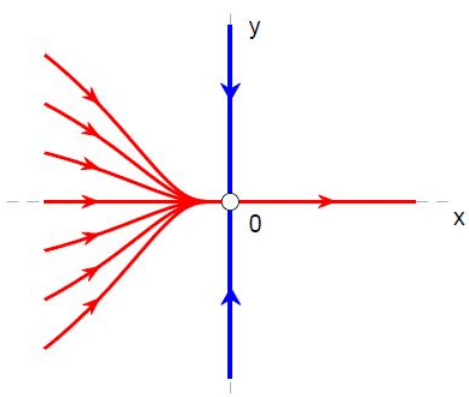


Figure 1. The phase diagram for the system of differential equations $x' = x^2,\ y' = -y$. Every red curve on the left coupled with the right $x$-axis is a local center manifold of the time-1 map of the solution operator at the fixed point $0$. There are infinitely many local center manifolds of the origin.


**Reference**: 1. A. Burchard, B. Deng, and K. Lu, *Smooth conjugacy of centre manifolds*, Procedings of the Royal Society of Edingurgh, 120A, pp.61–77, 1992.