Saddle-Node and Period-Doubling Bifurcations

Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $J = Df(\bar{q})$, and denote

$$\sigma^s = \sigma(J) \cap \{|z| < 1\}, \sigma^c = \sigma(J) \cap \{|z| = 1\}, \text{ and } \sigma^u = \sigma(J) \cap \{|z| > 1\}$$

the set of stable eigenvalues, center eigenvalues, unstable eigenvalues, respectively, of the linearizatoin $Df(\bar{q})$, counting multiplicity. Consider two cases

$$\text{either } \sigma^c = \{1\} \text{ or } \sigma^c = \{-1\}.$$

for which the former is called a saddle-node bifurcation problem and the latter a period-doubling bifurcation problem. Specifically, consider a one-parameter family of map $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$f(\bar{\lambda}, \bar{q}) = \bar{q}.$$

By augmenting the map to $\tilde{f}(\lambda, p) = (\lambda, f(\lambda, p))$, the parameter dimension is a trivial center direction for the augmented map. Then by the Center Manifold Theorem, for a sufficiently smooth f, the dynamics of $\tilde{f}$ is reduced to a two-dimensional local center manifold with λ being a trivial center dimension. The reduced map is just a parameterized one-dimensional mapping, denoted by $F(\lambda, x)$ for $(\lambda, x) \in \mathbb{R} \times \mathbb{R}$. Therefore, the dynamics of f near the fixed point $\bar{q}$ is completely determined by its center manifold reduction F. Here are application-friendly versions for these two elementary bifurcations.

Theorem 1 (Saddle-Node Bifurcation)**.** *Let $F(\lambda, x), (\lambda, x) \in \mathbb{R} \times \mathbb{R}$ be a C^2 function satisfying*

(i) $F(\bar{\lambda}, \bar{x}) = \bar{x}$ *for some* $\bar{\lambda}$ *and* $\bar{x}$ *for some* $\bar{\lambda}$ *and* $\bar{x}$.

(ii) $\frac{\partial F}{\partial x}(\bar{\lambda}, \bar{x}) = 1$.

(iii) $a := \frac{\partial^2 F}{\partial x^2}(\bar{\lambda}, \bar{x}) \neq 0$.

(iv) $b := \frac{\partial F}{\partial \lambda}(\bar{\lambda}, \bar{x}) \neq 0$.

Then

(a) If
$$ab(\lambda - \bar{\lambda}) = \frac{\partial^2 F}{\partial x^2}(\bar{\lambda}, \bar{x}) \cdot \frac{\partial F}{\partial \lambda}(\bar{\lambda}, \bar{x}) \cdot (\lambda - \bar{\lambda}) < 0,$$
then there are two fixed points of $F(\lambda, x)$ near $\bar{x}$ for λ near $\bar{\lambda}$, one is asymptotically stable, and the other is asymptotically unstable.

(b) If
$$ab(\lambda - \bar{\lambda}) = \frac{\partial^2 F}{\partial x^2}(\bar{\lambda}, \bar{x}) \cdot \frac{\partial F}{\partial \lambda}(\bar{\lambda}, \bar{x}) \cdot (\lambda - \bar{\lambda}) > 0,$$
then there is no fixed points for $F(\lambda, x)$.

Theorem 2 (Period-Doubling Bifurcation)**.** *Let $F(\lambda, x), (\lambda, x) \in \mathbb{R} \times \mathbb{R}$ be a C^3 function satisfying*

(i) $F(\bar{\lambda}, \bar{x}) = \bar{x}$ *for some* $\bar{\lambda}$ *and* $\bar{x}$ *for some* $\bar{\lambda}$ *and* $\bar{x}$.

(ii) $\frac{\partial F}{\partial x}(\bar{\lambda}, \bar{x}) = -1$.

(iii) $c := \frac{\partial^3 F^2}{\partial x^3}(\bar{\lambda}, \bar{x}) \neq 0$.

(iv) $d := \frac{d}{d\lambda}[\frac{\partial F}{\partial x}(\bar{\lambda}, x(\bar{\lambda}))] := \frac{d}{d\lambda}[\frac{\partial F}{\partial x}(\lambda, x(\lambda))]|_{\lambda=\bar{\lambda}} \neq 0$, *where* $x(\lambda)$ *is the fixed point of* $F(\lambda, x)$ *parameterized by* λ*:* $F(\lambda, x(\lambda)) = x(\lambda),\ x(\bar{\lambda}) = \bar{x}$.

Then

(a) If

$$cd(\lambda - \bar{\lambda}) = \frac{d}{d\lambda}[\frac{\partial F}{\partial x}(\bar{\lambda}, x(\bar{\lambda}))] \cdot \frac{\partial^3 F^2}{\partial x^3}(\bar{\lambda}, \bar{x}) \cdot (\lambda - \bar{\lambda}) > 0,$$

then there is a unique period-2 orbit of $F(\lambda, x)$ *near* $\bar{x}$ *for* λ *near* $\bar{\lambda}$*. Moreover, the period-2 orbit is asymptotically stable (respectively, unstable) if the corresponding fixed point* $x(\lambda)$ *is asymptotically unstable (respectively, stable).*

(b) If

$$cd(\lambda - \bar{\lambda}) = \frac{d}{d\lambda}[\frac{\partial F}{\partial x}(\bar{\lambda}, x(\bar{\lambda}))] \cdot \frac{\partial^3 F^2}{\partial x^3}(\bar{\lambda}, \bar{x}) \cdot (\lambda - \bar{\lambda}) < 0,$$

then there is no period-2 orbits near $\bar{x}$*.*

Remark: For $F(\lambda, x(\lambda)) = x(\lambda)$, by implicit differentiation,

$$\frac{dx}{d\lambda}(\lambda) = \frac{\partial F}{\partial \lambda}(\lambda, x(\lambda))/[1 - \frac{\partial F}{\partial x}(\lambda, x(\lambda))] \text{ and } \frac{dx}{d\lambda}(\bar{\lambda}) = \frac{\partial F}{\partial \lambda}(\bar{\lambda}, \bar{x})/2.$$

As a result,

$$\frac{d}{d\lambda}[\frac{\partial F}{\partial x}(\bar{\lambda}, x(\bar{\lambda}))] = \frac{\partial^2 F}{\partial \lambda \partial x}(\bar{\lambda}, \bar{x}) + \frac{\partial^2 F}{\partial x^2}(\bar{\lambda}, \bar{x})\frac{\partial F}{\partial x}(\bar{\lambda}, \bar{x})/2.$$

To keep the presentation simple enough, we will present a proof for an equivalent form of the second theorem. Similar treatment can be applied to the saddle-node bifurcation.