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Smooth conjugacy of centre manifolds

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Synopsis

In this paper, we prove that for a system of ordinary differential equations of class $C^{r+1,1}$, $r \geqq 0$ and two arbitrary $C^{r+1,1}$ local centre manifolds of a given equilibrium point, the equations when restricted to the centre manifolds are C^r conjugate. The same result is proved for similinear parabolic equations. The method is based on the geometric theory of invariant foliations for centre-stable and centre-unstable manifolds.

1. Introduction

Following the pioneering work on invariant manifold theory by Poincaré [**33**], Lyapunov [**23**], Hadamard [**16**], Perron [**31**], the theory of centre manifolds for finite dimensional dynamical systems has been developed by Pliss [**32**], Kelley [**22**], Hirsch, Pugh and Shub [**20, 21**], Fenichel [**12, 13**], Wells [**38**], Carr [**39**], Chow and Hale [**3**], Sijbrand [**35**], Vanderbauwhede [**36**] and others. Centre manifold theory for infinite dimensional systems has been studied by Henry [**19**], Carr [**39**], Hale, Magalhães and Oliva [**18**], Hale and Lin [**17**], Mielke [**27**], Bates and Jones [**2**], Chow and Lu [**6, 7**], Chow, Lin and Lu [**5**], Vanderbauwhede and Iooss [**37**], and many others.

Although most fundamental problems in the theory of centre manifolds have been solved, a convincing solution for the uniqueness problem of local centre manifolds has been missing. In the context of ordinary differential equations, the standard method for constructing a local centre manifold at a given equilibrium point is to extend the locally defined equation by a cut-off function to a globally defined one for which existence and smoothness of a unique global centre manifold can be established by either Hadamard's or Perron's method (cf. e.g. Anosov [**1**], Hirsch, Pugh and Shub [**21**]). The nonuniqueness of local centre manifolds results from the use of arbitrary cut-off functions in the construction as shown by Sijbrand [35], and under certain conditions there is an exceptional case given by Bates and Jones [2]. There is little doubt that the dynamics on different local centre manifolds should behave in the same way, but the question is in what sense or to what degree that is so. This question has attracted a good deal of attention in the literature since the birth of the theory, and some results, by no means a complete account here, can be summarised as follows: (1) any local centre manifold of a given equilibrium point must contain all the invariant sets, such as equilibrium points, periodic, homoclinic, heteroclinic orbits, etc. near the equilibrium point; (2) the formal Taylor expansions at the equilibrium point of the vector field when restricted to different local centre manifolds are exactly the same (see, e.g. Carr [39], Sijbrand [35]). In this paper, however, we examine the uniqueness question from the standpoint of smooth conjugacy. More specifically, we want to show that the restrictions of the equation to two arbitrary local centre manifolds are actually topologically or differentiably conjugate, depending on the smoothness of the vector field. That is, the smooth conjugacy class of the restricted equations is indeed unique.

Note that, speaking at the conjugacy level, our result does include the properties (1, 2) above, but the converse is less clear. Indeed, we still do not know what these properties imply about the conjugacy problem within the context of centre manifold theory, and nothing at all can be said about it outside the theory. For example, for the two equations $\dot{x} = -xe^{-1/x_2}$ and $\dot{x} = xe^{-1/x_2}$, the origin is the only invariant set and the formal Taylor expansions at the origin are the same. But they are not conjugate because the equilibrium point is stable for the first equation and unstable for the second one.

The main result, Theorem 2.1, is treated in terms of two types of differential equations: ordinary differential equations and semilinear parabolic equations. The method is based on invariant foliations for centre-stable and centre-unstable manifolds. The theory used here is twofold. Firstly, due to many people's studies of this subject, cf., e.g. [1, 12, 13, 21] for finite dimensional systems, and [5] for infinite dimensional systems, it has become a standard procedure to construct simultaneously for a sufficiently differentiable dynamical system a local centre, centre-stable, and centre-unstable manifold, together with a stable and an unstable foliation on the centre-stable and centre-unstable manifold, respectively (see Theorems 5.1.5.2). Secondly, we show in Theorem 3.1 that these geometric structures can be recovered for a given local centre manifold by extending the local manifold of the locally defined equation to the global centre manifold of a globally defined equation for which the invariant manifold and foliation structures follow easily from known results. Based on this, our geometrical proof is carried out in two steps. We first show in Lemma 4.1 that if two local centre manifolds happen to share a common centre-stable or a common centre-unstable manifold, then the conjugacy follows from the foliation of that manifold. In general, two centre manifolds lie in neither a common centre-stable nor a common centre-unstable manifold (a simple example is given in Section 3). In this case, we show that the flow structures on the two manifolds can be transformed from one to the other through the flow structure on a third local centre manifold which lies in the intersection of a centre-stable manifold containing one of the given centre manifolds and a centre-unstable manifold containing the other.

Recently the theory of inertial manifolds has been developed for some dissipative evolution equations, see [**15**, **26**, **7**, **9**]. We find that in this context a similar conjugacy question arises. An inertial manifold of such a system is a finite-dimensional invariant manifold which attracts solutions exponentially. In particular, this implies that it contains the global attractor. Under certain conditions, such as a spectral gap condition or the cone condition, inertial manifolds do exist. Indeed, similar to the construction of local invariant manifolds, one obtains such an inertial manifold by modifying the equation outside an absorbing ball, which contains the attractor, with a cut-off function and constructing an inertial manifold for the modified equation. Restricting that globally-defined inertial manifold to the absorbing ball gives the desired inertial manifold for the original system that is also referred to as a local inertial manifold. Again, different cut-off functions give rise to different local inertial manifolds; and it is natural to ask if the dynamics on different local inertial manifolds are the same up to smooth conjugacy. In fact, we shall be able to conclude at the end of this paper that the same answer with C^0 or C^1 conjugacy also applies.

We remark that invariant foliation theory has been applied to conjugacy problems by many people. For example, Anosov [**1**], Palis [**28**], Palis and Smale [**29**], and Robinson [**34**] used it to analyse structural stability of finite-dimensional dynamical systems. Palis and Takens [**30**] used it to prove a result which implies that two differential equations are locally topologically conjugate if the equations when restricted to their centre manifolds are topologically congugate. Lu [**24**, **25**] used the infinite-dimensional counterpart to generalise the Hartman-Grobman theorem to parabolic equations. It is also very useful in other areas of study of dynamical systems. In fact, it is one of the key components for the geometric theory of singular perturbations of Fenichel [**14**] and its applications, cf. e.g. Deng [**11**]. It also plays an important role in the theory of homoclinic and heteroclinic bifurcations of Chow and Lin [**4**] and Deng [**10**].

The paper is organised as follows. In Section 2, we state the main result. In Section 3, we introduce the principal tool, namely, local invariant foliation theory. The main result, Theorem 2.1, is proved in Section 4. The existence theorem for local invariant foliations from Section 3 is proved in Sections 5 and 6. We end the paper in Section 7 with a discussion of some variations and possible generalisations of our results.

2. The main result

Let $\mathbf{X}$ be a Banach space and consider a semilinear evolution equation

$$\dot{x} = Ax + f(x), \quad x \in \mathbf{X}, \tag{2.1}$$

together with two hypotheses:

(2.1a) A is a sectorial operator from a dense domain $D(A) \subset \mathbf{X}$ into $\mathbf{X}$. Let $\sigma(A)$ denote the spectrum of A, then $\sigma(A) \cap \{\lambda \mid \operatorname{Re} \lambda \geqq 0\}$ consists of a finite number of isolated eigenvalues, each with a finite-dimensional generalised eigenspace.

(2.1b) Let $\mathbf{X}^\alpha$ be the fractional power space associated with the operator A and $U \subset \mathbf{X}^\alpha$ be a neighbourhood of the origin $x = 0$. Then f: $U \to \mathbf{X}$ is of class $C^{r+1,1}$ with $r \geqq 0$. Moreover, f has zero of higher order at the origin, i.e., $f(0) = 0$ and the linearisation $Df(0) = 0$.

For background information concerning semilinear evolution equations, we refer to **[19]**. We only point out that when $\mathbf{X}$ is finite-dimensional, A is just a matrix, $\mathbf{X}^\alpha = \mathbf{X}$ for all α and equation (2.1) is simply a system of ordinary differential equations. It is known that if $f \in C^{0,1}(U, \mathbf{X})$, for every $x_0 \in U \subset \mathbf{X}^\alpha$, unique solution $x(t) \in D(A)$ for small $t > 0$ with $x(0) = x_0$. Let $S_t(x_0) = x(t)$ denote the resulting local flow, then a submanifold $W \subset \mathbf{X}^\alpha$ is said to be locally invariant if for every $x \in W$ there is a curve in W that is a solution of the equation and contains x as an interior point.

Let $E^c \subset \mathbf{X}$ be the generalised finite-dimensional eigenspace corresponding to the eigenvalues $\sigma^c := \sigma(A) \cap \{\lambda \mid \operatorname{Re} \lambda = 0\}$. Then a locally invariant differentiable manifold $W^c \subset U$ is said to be a local centre manifold of the equilibrium point $x = 0$ if W^c is tangent to E^c at the origin $x = 0$.

For simplicity, by a C^r diffeomorphism we mean a homeomorphism if $r = 0$. Our main result is the following theorem:

THEOREM 2.1. *Assume the hypotheses* (2.1*a*, *b*) *for equation* (2.1). *Then the local flows on two arbitrary* $C^{r+1,1}$ *local centre manifolds in* $U \subset \mathbf{X}^\alpha$ *are locally* C^r *conjugate. More specifically, if* W_1^c *and* W_2^c *are such manifolds, then there is a neighbourhood* $V \subset U$ *of the origin and a* C^r *diffeomorphism* ϕ: $W_1^c \cap V \to W_2^c \cap V$ *such that*

$$S_t \circ \phi(x) = \phi \circ S_t(x)$$

for all $x \in W_1^c \cap V$ *and all* $t > 0$ *so long as* $S_t(x) \in W_1^c \cap V$.

Two different versions of this theorem with weaker regularity assumptions on the nonlinear term f and slightly more restrictive assumptions on the centre manifolds are given in Section 7.

3. Local foliations

Let $\sigma^s := \sigma(A) \cap \{\lambda \mid \operatorname{Re}\lambda < 0\}$, $\sigma^u := \sigma(A) \cap \{\lambda \mid \operatorname{Re} \lambda > 0\}$. Then $\sigma(A) = \sigma^s \cup \sigma^c \cup \sigma^u$. By hypothesis (2.1a), σ^u consists of a finite number of isolated eigenvalues, each with a finite-dimensional generalised eigenspace. Let E^u be the generalised eigenspace corresponding to σ^u in $D(A)$. Because both E^c and E^u are finite-dimensional in $D(A)$, the projection π_i: $D(A) \to E^i \subset D(A)$ can be continuously extended to π_i: $\mathbf{X} \to E^i \subset \mathbf{X}$ for $i = c, u$. Therefore, $\pi_s := \mathrm{ID}_X - \pi_c - \pi_u$ is also a projection map, and so we can denote $E^s := \pi_s(\mathbf{X})$. We denote throughout

$$\mathbf{X}^\alpha = \{E^s \cap \mathbf{X}^\alpha\} \oplus E^c \oplus E^u, \quad E^{cs} := \{E^s \cap \mathbf{X}^\alpha\} \oplus E^c, \quad E^{cu} := E^c \oplus E^u.$$

Note that $E^i \subset \mathbf{X}^\alpha$ for all $0 \leqq \alpha$, $i = c, u$, whereas E^{cs} depends on the fractional power α. We also use $x = x_s + x_c + x_u$ with $x_i \in E^i$, $i = s, c, u$ and $\|x\|_\alpha = \|x_s\|_\alpha + \|x_c\| + \|x_u\|$, where $\|\cdot\|_\alpha$ is the graph norm for $\mathbf{X}^\alpha$ and $\|\cdot\|$ is the norm for $\mathbf{X}$.

Let $V \subset U$ be a neighbourhood of the origin where $U \subset \mathbf{X}^\alpha$ is as in the hypothesis (2.1b). A locally invariant manifold $W^{cs} \subset V$ (respectively $W^{cu} \subset V$) is said to be a local centre-stable (respectively centre-unstable) manifold of the equilibrium point $x = 0$ if it is tangent to E^{cs} (respectively E^{cu}) at this point when differentiable, or if it is the graph of a $C^{0,1}$ function h: $E^{cs} \cap V \to E^u$ (respectively h: $E^{cu} \cap V \to E^s$) with $h(0) = 0$ and sufficiently small Lipschitz constant Lip $h < 1$. We shall take Lip $h < \frac{1}{3}$ in various places for technical reasons. For simplification, we also use Lip $W^{cs} < \rho$ to mean Lip $h < \rho$. Such slightly abused notation also applies to centre-unstable and centre manifolds.

Let $W \subset V$ be a locally invariant manifold of the origin. Let $\{\mathscr{F}(p) \mid p \in W\}$ be a family of submanifolds of W parametrised by $p \in W$. $\{\mathscr{F}(p) \mid p \in W\}$ is said to be locally positively (respectively negatively) invariant if $S_t(\mathscr{F}(p)) \cap V \subset \mathscr{F}(S_t(p)) \cap V$ for those $t \geqq 0$ (respectively $t \leqq 0$, provided $S_t(p)$ is well-defined on W) with $S_\tau(p) \in W \cap V$ for all $\tau \in [0, t]$ (respectively $\tau \in [t, 0]$). It is called a C^r family of $C^{r,1}$ manifolds if the set $\{(p, q) \mid p \in W, q \in \mathscr{F}(p)\}$ is a $C^r \times C^{r,1}$ submanifold of $\mathbf{X}^\alpha \times \mathbf{X}^\alpha$, where $r \geqq 0$.

Let W^{cs} be a local centre-stable manifold in neighbourhood V of the origin. A family of submanifolds $\{\mathscr{F}^s(p) \mid p \in W^{cs}\}$ is said to be a $C^r \times C^{r,1}$ stable foliation for W^{cs} if the following conditions are satisfied:

(i) $p \in \mathscr{F}^s(p)$ of each $p \in W^{cs}$;

(ii) $\mathscr{F}^s(p)$ and $\mathscr{F}^s(q)$ are disjoint or identical for each p and q in W^{cs};

(iii) if $r \geqq 1$, $\mathscr{F}^s(0)$ is tangent to E^s at the origin; or if $r = 0$, every leaf $\mathscr{F}^s(p)$ is the graph of a $C^{0,1}$ function, say $\varphi(p, \cdot)$: $E^s \cap V \to E^{cu} \cap V$ with Lipschitz constant Lip $\varphi(p, \cdot) < 1$ for all $p \in W^{cs}$ near the origin;

(iv) $\{\mathscr{F}^s(p) \mid p \in W^{cs}\}$ is a positively invariant C^r family of $C^{r,1}$ manifolds for W^{cs}.

Note that we can always identify the local centre-stable manifold W^{cs} with the linear space E^{cs} locally through a function from E^{cs} into E^u whose graph is the manifold itself. Thus, in terms of the coordinate system for E^{cs}, conditions (i–iv) amount to saying that there is a $C^r \times C^{r,1}$ function φ: $\{E^{cs} \cap V\} \times \{E^s \cap V\} \to E^c$ with $\mathscr{F}^s(p) =$ graph $\varphi(p, \cdot)$ that satisfies (i) $p_c = \varphi(p, p_s)$; (ii) $p_c = \varphi(q, p_s)$ if and only if $q_c = \varphi(p, q_s)$; (iii) either the partial derivative with respect to the second variable $D_2\varphi(0, 0) = 0$ or Lip $\varphi(p, \cdot) < 1$, depending on whether $r > 0$ or $r = 0$; and (iv) $S_t(x_s + \varphi(p, x_s)) = \pi_s S_t(x) + \varphi(S_t(p), \pi_s S_t(x))$for small $t \geqq 0$, where $p = p_s + p_c$, $q = q_s + q_c \in E^{cs} \cap V$.

Similarly, a $C^r \times C^{r,1}$ unstable foliation $\{F^u(p) \mid p \in W^{cu}\}$ for a local centre-unstable manifold W^{cs} satisfies:

(i) $p \in \mathscr{F}^u(p)$ for each $p \in W^{cu}$;

(ii) $\mathscr{F}^u(p)$ and $\mathscr{F}^u(q)$ are disjoint or identical for each p and q in W^{cu};

(iii) if $r \geqq 1$, $\mathscr{F}^u(0)$ is tangent o E^u at the origin; or if $r = 0$, it is the graph of a $C^{0,1}$ function, say $\psi(p, \cdot)$: $E^u \cap V \to E^{cs} \cap V$ with the property that Lip $\psi(p, \cdot) < 1$ for all $p \in W^{cu}$ near the origin;

(iv) $\{\mathscr{F}^u(p) \mid p \in W^{cu}\}$ is a negatively invariant C^r family of $C^{r,1}$ manifolds for W^{cu}.

Note that an unstable foliation can be expressed in terms of the local coordinate system for E^{cu} in the same way as was done for stable foliations in the previous paragraph.

We remark that the formulation of the stable and unstable foliations in this paper follows that of Fenichel **[14]** for the geometric theory of singular perturbations.

One of the key ingredients for proving the main result is the following theorem which will be proved in Section 6:

THEOREM 3.1. *Assume the hypotheses* (2.1*a*, *b*) *for equation* (2.1). *Then for any* $C^{r+1,1}$ *local centre manifold* $W^c \subset U$ *of the origin, there are a* $C^{r,1}$ *local centre-stable manifold* W^{cs} *and a* $C^{r,1}$ *local centre-unstable manifold* W^{cu} *in a neighbourhood* $V \subset U$ *of the origin both containing* W^c *as a submanifold and satisfying* $\text{Lip}\, W^i < \frac{1}{3}$, $i = cs, cu$. *Moreover, there are a* $C^r \times C^{r,1}$ *stable foliation on* W^{cs} *and a* $C^r \times C^{r,1}$ *unstable foliation on* W^{cu}.

We end this section with an example. Consider the three-dimensional system of ordinary differential equations: $\dot{x} = -x$, $\dot{y} = y^2$, $\dot{z} = z$. Every local centre manifold of the origin is given by one of the curves $W^e := \{(c_1 e^{1/y}, y, 0) \mid \text{for small } y \leqq 0\} \cup \{(0, y, c_2 e^{-1/y}) \mid \text{for small } y \geqq 0\}$ for some choice of the constants c_1, c_2. Moreover, given a local centre manifold W^c, there is a local centre-stable manifold W^{cs} containing W^c and, in terms of a stable foliation, $W^{cs} = \{p + (1, 0, 0)x \mid p \in W^c, |x| \ll 1\}$. Similarly, a local centre-unstable manifold W^{cu} containing W^c can be expressed as $\{p + (0, 0, 1)z \mid p \in W^c, |z| \ll 1\}$ in terms of an unstable foliation. In fact, every local centre stable manifold is given by $z = 0$ for $y \leqq 0$, $z = e^{-1/y} f(e^{-1/y} x)$ for $y > 0$, x, y small, where f is any sufficiently smooth function; similarly, every local centre unstable manifold is given by $x = e^{1/y} f(e^{1/y} z)$ for $y < 0$, $x = 0$ for $y \geqq 0$, y, z small, g a sufficient smooth function. A given local centre manifold lies on a certain centre stable (centre unstable) manifold, if $c_2 = f(0)$ ($c_1 = g(0)$, respectively). Observe that two pairs of distinct constants $(c_., c_2)$ can be chosen so that the resulting local centre manifolds share neither a common centre-stable manifold nor a common centre-unstable manifold. This observation motivates the proof of the main theorem in the next section.

4. Proof of Theorem 2.1

LEMMA 4.1. *Let* $W^{cs} \subset \mathbf{X}^\alpha$ *be a* $C^{r,1}$ *local centre-stable manifold of the origin with* $\text{Lip}\, W^{cs} < 1$ *and* $r \geqq 0$. *Assume there is a* $C^r \times C^{r,1}$ *stable foliation on* W^{cs}. *Then for two arbitrary* $C^{r,1}$ *local centre manifolds* $W_1^c \subset W^{cs}$, $W_2^c \subset W^{cs}$ *of the origin with* $\text{Lip}\, W_i^c < 1$, $i = 1, 2$ *the conclusion of Theorem* 2.1 *holds true. That is, there is a neighbourhood* $V \subset \mathbf{X}^\alpha$ *of the origin and a* C^r *diffeomorphism* $\phi: W_1^c \cap V \to W_2^c \cap V$ *such that*

$$S_t \circ \phi(x) = \phi \circ S_t(x)$$

for all $x \in W_1^c \cap V$ *and all* t *satisfying* $S_t(x) \in W_1^c \cap V$.

Proof. We only demonstrate the $C^{0,1}$ case since the $C^{r,1}$ case with $r > 0$ is simplified when the contraction mapping principle is replaced by the implicit function theorem.

By assumption, let $\{F^s(p) \mid p \in W^{cs}\}$ be the local stable foliation in a neighbourhood U of the origin. We want to show that a homeomorphism is defined by $\phi(p) = q := \mathscr{F}^s(p) \cap W_2^c$ for $p \in W_1^c$ near the origin.

To do this, we begin by identifying the local centre-stable manifold W^{cs} with the coordinate plane $E^{cs} \cap U$ via a $C^{0,1}$ function whose graph is the manifold W^{cs}. Now, in terms of the coordinate system for E^{cs}, let $W_i^c = \text{graph } h_i$ for some $C^{0,1}$ function h_i: $E^c \cap U \to E^s \cap U$ with $\text{Lip } h_i < 1$ and let φ: $\{E^{cs} \cap U\} \times \{E^s \cap U\} \to E^c \cap U$ represent the stable foliation on W^{cs}, satisfying $\mathscr{F}^s(p) = \text{graph } \varphi(p, \cdot)$, and $\text{Lip } \varphi(p, \cdot) < 1$ for all $p \in E^{cs} \cap U$. Let $\delta > 0$ be so small that the closed box $B := \{x \in E^{cs} \mid \|x_s\|_\alpha \leqq \delta, \|x_c\| \leqq \delta\}$ centred at the origin is contained entirely in U. Consider the operator $\bar{x} = \Phi(p, x)$, $x \in B$ defined by $\bar{x}_c = \varphi(p, x_s)$, $\bar{x}_s = h_2(x_c)$ and parametrised by $p \in W_1^c \cap U$. We want to show that for some carefully chosen neighbourhood $V \subset B$ of the origin, the fixed point $q \in W_2^c \cap V$, which is the intersection point of $\mathscr{F}^s(p)$, $p \in W_1^c \cap V$ and W_2^c, gives rise to the conjugating map ϕ.

To be precise, choose a neighbourhood $V_0 \subset B$ of the origin so that $\|\varphi(p, x_s)\| \leqq \delta$ for all $p \in V_0$, $\|x_s\|_\alpha \leqq \delta$ and $\|h_i(x_c)\|_\alpha \leqq \delta$ for all $\|x_c\| \leqq \delta$, $i = 1, 2$. This can be done because we have the strict inequalities $\text{Lip } \varphi(p, \cdot) < 1$ and $\text{Lip } h_i < 1$. Hence, $\Phi(p, \cdot)$ maps B into itself for all $p \in W_1^c \cap V_0$. Moreover, by the box norm for the space $\mathbf{X}$, we have

$$\text{Lip } \Phi(p, \cdot) = \max(\text{Lip } \varphi(p, \cdot), \text{Lip } h_1\} < 1$$

uniformly for all $p \in W_1^c \cap V_0$. Therefore, by the contraction mapping principle, there is a unique fixed point $q(p) \in W_2^c \cap B$ for every $p \in W_1^c \cap V_0$. Denote by $q := \phi(p)$ the fixed point, then ϕ is C^0. Arguing symmetrically, we can also show that for every point $q \in W_2^c \cap V_0$ there is a unique fixed point $p \in W_1^c \cap B$ of the operator $\bar{x}_c = \varphi(q, x_s)$, $\bar{x}_s = h_1(x_c)$ so that the fixed point $p := \bar{\phi}(q)$, which is the intersection point of $\mathscr{F}^s(q)$ and $W_1^c \cap B$, depends continuously on $q \in W_2^c \cap V_0$.

We now claim that the function ϕ is actually locally invertible with inverse $\bar{\phi}$. More specifically, we claim

$$(\phi \mid_{\bar{\phi}(W_2^c \cap V_0) \cap V_0})^{-1} = \bar{\phi} \mid_{\phi(W_1^c \cap V_0) \cap V_0}.$$

In fact, let $p = \bar{\phi}(q)$ with $q \in \phi(W_1^c \cap V_0) \cap V_0$. We first need to show $q = \phi(p)$ with $p \in \bar{\phi}(W_2^c \cap V_0) \cap V_0$. By definition, we have $p_c = \varphi(q, p_s)$, $p_s = h_1(p_c)$ with $q_s = h_2(q_c)$. Because of the foliation property (ii), $q_c = \varphi(p, q_s)$, so q is the fixed point of the operator $\Phi(p, \cdot)$, provided that $p \in W_1^c \cap V_0$. Suppose on the contrary that $p \in B - V_0$. By definition, there exists for $q \in \phi(W_1^c \cap V_0) \cap V_0$ a point $\tilde{p} \in W_1^c \cap V_0$ such that $q = \phi(\tilde{p})$, i.e. $q_c = \varphi(\tilde{p}, q_s)$, $q_s = h_2(q_c)$. By the foliation property (ii) again, $\tilde{p}_c = \varphi(q, \tilde{q}_s)$. This gives rise to the following contradiction:

$$\|p_c - \tilde{p}_c\| = \|\varphi(q, p_s) - \varphi(q, \tilde{p}_s)\| < \|p_s - \tilde{p}_s\|_\alpha$$

and

$$\|p_s - \tilde{p}_s\|_\alpha = \|h_1(p_c) - h_1(\tilde{p}_c)\|_\alpha < \|p_c - \tilde{p}_c\|,$$

because of $\tilde{p}_s = h_1(\tilde{p}_c)$. This implies $\|p - \tilde{p}\|_\alpha < \|p - \tilde{p}\|_\alpha$. Therefore, we must have $p = \tilde{p} \in W_1^c \cap V_0$, $q = \phi(p) = \phi \circ \bar{\phi}(q)$. Conversely, the same argument shows that if $q = \phi(p)$ with $p \in \bar{\phi}(W_2^c \cap V_0) \cap V_0$, then $p = \bar{\phi}(q) = \bar{\phi} \circ \phi(p)$. This proves the claim.

By the foregoing argument the neighbourhood V can be any open set satisfying

$$V \cap W_2^c = \phi(W_1^c \cap V_0) \cap V_0 \quad \text{and} \quad V \cap W_1^c = \bar{\phi}(W_2^c \cap V_0) \cap V_0,$$

for instance, we can take $V := V_0 - \{(W_2^c - \phi(W_1^c \cap V_0) \cap V_0) \cup (W_1^c - \bar{\phi}(W_2^c \cap V_0) \cap V_0)\}$.

Finally, since $S_t(p) \in W_1^c$, $S_t(\phi(p)) \in W_2^c$, $\mathscr{F}^s(\phi(p)) = \mathscr{F}^s(p)$ and $S_t(\mathscr{F}^s(p)) \subset \mathscr{F}^s(S_t(p))$, locally, by the invariance of the centre manifolds and the foliation, we have

$$S_t \circ \phi(p) = S_t(\mathscr{F}^s(p) \cap W_2^c) = \mathscr{F}^s(S_t(p)) \cap W_2^c = \phi \circ S_t(p)$$

so long as $S_t(p) \in W_1^c \cap V$. □

We remark that the same conclusion also holds true for any $C^{r,1}$ local centre-unstable manifold together with a $C^r \times C^{r,1}$ unstable foliation.

LEMMA 4.2. *Let W_1^c, W_2^c be two $C^{r+1,1}$ local centre manifolds of the origin. Let $W^{cs} \subset \mathbf{X}^\alpha$ be a $C^{r,1}$ local centre-stable manifold containing W_1^c and $W^{cu} \subset \mathbf{X}^\alpha$ a $C^{r,1}$ local centre-unstable manifold containing W_2^c, both constructed according to* Theorem 3.1. *Then the intersection $W^{cs} \cap W^{cu}$ is another $C^{r,1}$ local centre manifold, W^c, of the origin with* $\operatorname{Lip} W^c < 1$.

Proof. For the same reason as in the last proof, we only demonstrate the $C^{0,1}$ case. Since the centre-stable and centre-unstable manifolds are constructed according to Theorem 3.1, they satisfy $W^{cs} = \operatorname{graph} h^{cs}$, $W^{cu} = \operatorname{graph} h^{cu}$ for some $C^{0,1}$ functions h^{cs}, h^{cu} defined near the origin with $\operatorname{Lip} h^i < \frac{1}{3}$, $i = cs, cu$. The intersection $W^{cs} \cap W^{cu}$ consists of all points $x_s + x_c + x_u \in \mathbf{X}^\alpha$ satisfying $x_u = h^{cs}(x_c, x_s)$, $x_s = h^{cu}(x_c, x_u)$. Think of the right-hand side of these equations as an operator parametrised by x_c, then the contraction constant of this operator is bounded by $\max\{\operatorname{Lip} h^{cs}(x_c, \cdot), \operatorname{Lip} h^{cu}(x_c, \cdot)\} < 1$ uniformly for all small $\|x_c\|$. By the contraction mapping principle, one can solve uniquely for a C^0 function $x_s + x_u = h^c(x_c)$ for small $\|x\|_\alpha$. h^c is Lipschitz with

$$\operatorname{Lip} h^c \leqq \frac{\operatorname{Lip} h^{cs} + \operatorname{Lip} h^{cu}}{1 - \max\{\operatorname{Lip} h^{cs}, \operatorname{Lip} h^{cu}\}} < 1$$

as $\operatorname{Lip} h^i < \frac{1}{3}$, $i = cs, cu$ by Theorem 3.1. This sketch can be made as precise as we have done for the proof of Lemma 4.1 above.

To show $W^c := \operatorname{graph} h^c$ is indeed a $C^{0,1}$ local centre manifold of the origin, it suffices to show that it is locally invariant. For $\mathbf{X}$ finite dimensional, this is trivial. Otherwise, we proceed as follows: Let $p \in W^c$, and let $x(t) \in W^{cu}$ be the solution curve in the centre-unstable manifold containing p as an interior point. Because of the uniqueness for the initial value problem of equation (2.1) and the invariance of W^{cs}, $x(t) \in W^{cs}$ for small $t \geqq 0$. Because any backward extension of the solution $x(t)$ must also be in W^{cs} by the remark after the proof of Theorem 3.1 in Section 6, we have that $x(t) \in W^{cs}$ for all small $|t|$ for which it is defined. Hence, $x(t) \in W^c = W^{cs} \cap W^{cu}$ for the same small $|t|$. □

Proof of Theorem 2.1. Let W^{cs} be a $C^{r,1}$ local centre-stable manifold containing W_1^c and W^{cu} be a $C^{r,1}$ local centre-unstable manifold containing W_2^c by Theorem 3.1. Then $W_3^c := W^{cs} \cap W^{cu}$ is another $C^{r,1}$ local centre manifold by Lemma 4.2. By Theorem 3.1 again, there exist $C^r \times C^{r,1}$ stable and unstable foliations on W^{cs} and W^{cu}, respectively. Hence, the conditions of Lemma 4.1 are satisfied for W_1^c, W_3^c on W^{cs}, and W_2^c, W_3^c on W^{cu}, respectively. Hence, the flows on W_i^c and W_3^c are C^r conjugate for $i = 1, 2$. Therefore, the local flows on W_1^c and W_2^c are C^r conjugate because C^r conjugacy is an equivalence relation. □

5. Global foliations

The following two sections are dedicated to the proof of Theorem 3.1. The idea of the proof is to show that the local result can be obtained by extending the given local centre manifold of the locally defined equation to the global centre manifold of a globally defined equation to which some modified global result applies. In this section, we introduce the global theory of invariant manifolds and foliations.

We begin with some more notation. For $\delta > 0$, define $N_\delta := \{x \in \mathbf{X}^\alpha \mid \|x_s\|_\alpha < \delta, \|x_u\| < \delta\}$, a tubular neighbourhood of E^c. Also, $E_\delta^s := E^s \cap N_\delta$, $E_\delta^u := E^u \cap N_\delta$, $E_\delta^{cs} := E^{cs} \cap N_\delta$, $E_\delta^{cu} := E^{cu} \cap N_\delta$, and $E_\delta^{su} := E^{su} \cap N_\delta$ where $E^{su} := E^s \oplus E^u$. For simplicity, we denote $\pi_{su} := \pi_s + \pi_u$, $A_{su} := A\pi_{su}$, $x_{su} := \pi_{su}x = x_s + x_u$.

In addition to equation (2.1) which is only defined near the origin, we consider equations of the form

$$\dot{x} = Ax + F(x), \tag{5.1}$$

where $F(x) \in \mathbf{X}$ is defined for all $x \in \mathbf{X}^\alpha$ for some $0 \leqq \alpha < 1$. From now on, we use W_{loc}^c, W_{loc}^{cs}, W_{loc}^{cu} for local centre manifold, local centre-stable manifold, local centre-unstable manifold, respectively; while W^c, W^{cs}, W^{cu} are reserved for global centre manifold, global centre-stable manifold, global centre-unstable manifold, respectively. The former are defined as in the previous sections and the latter are defined as follows:

$$\begin{aligned} W^c &:= \left\{x \in \mathbf{X}^\alpha \,\middle|\, \sup_{|t|<\infty} \|\pi_{su}S_t(x)\|_\alpha < \infty\right\}, \\ W^{cs} &:= \left\{x \in \mathbf{X}^\alpha \,\middle|\, \sup_{0\leqq t<\infty} \|\pi_u S_t(x)\|_\alpha < \infty\right\}, \\ W^{cu} &:= \left\{x \in \mathbf{X}^\alpha \,\middle|\, \sup_{-\infty<t\leqq 0} \|\pi_s S_t(x)\|_\alpha < \infty\right\}. \end{aligned} \tag{5.2}$$

We have the following results:

THEOREM 5.1 (Existence, uniqueness and smoothness of global invariant manifolds). *Assume hypothesis* (2.1*a*) *for the linear operator A and that* $F \in C^{0,1}(\mathbf{X}^\alpha, \mathbf{X}) \cap C^{r,1}(N_\delta, \mathbf{X})$ (*respectively* $C^{0,1}(\mathbf{X}^\alpha, \mathbf{X}) \cap C^r(N_\delta, \mathbf{X})$) *satisfying* $F(0) = 0$, *where* $0 \leqq \alpha < 1$, $r \geqq 0$. *There exist constants* $m = O(\delta)$, $\epsilon = \epsilon(r)$ and $\delta_0 < \delta$ *such that if* $\sup_{x\in\mathbf{X}^\alpha} \|\pi_{su}F(x)\| < m$, Lip $F < \epsilon$, *there exist for equation* (5.1) *a unique global centre-stable manifold and a unique global centre-unstable* manifold. These manifolds are of class $C^{r,1}$ (respectively C^r) *and are given by* $W^{cs} = \text{graph } h^{cs}$, $W^{cu} = \text{graph } h^{cu}$. Here, $h^{cs}: E^{cs} \to E^u_{\delta_0}$, $h^{cu}: E^{cu} \to E^s_{\delta_0}$ *are of class* $C^{0,1}$, *and* $h^{cs}|_{E^{cs}_{\delta_0}}$, $h^{cu}|_{E^{cu}_{\delta_0}}$ *are of class* $C^{r,1}$ *(respectively* C^r*), satisfying* $h^i(0) = 0$, $\text{Lip } h^i < \frac{1}{3}$, *or* $Dh^i(0) = 0$ *for* $i = cs, cu$, *if* $DF(0) = 0$ *for* $r \geqq 1$. *Furthermore, there is a unique global centre manifold given by* $W^c = W^{cs} \cap W^{cu} = \text{graph } h^c$, *where* $h^c: E^c \to E^{su}_{\delta_0}$ *is of class* $C^{r,1}$ *(respectively* C^r*) satisfying* $h^c(0) = 0$, $\text{Lip } h^c < 1$ *and for* $r \geqq 1$, $Dh^c(o) = 0$ *if* $DF(0) = 0$.

THEOREM 5.2 (Existence and smoothness of global foliations). *Assume the conditions of Theorem* 5.1. *Then there exist constants* $m = O(\delta)$, $\epsilon = \epsilon(r)$ *and* $\delta_0 < \delta$ *such that if* $\|\pi_{su}F\| < m$, $\text{Lip } F < \epsilon$, *there exist for equation* (5.1) *unique global centre-stable and centre-unstable manifolds* W^{cs}, W^{cu} *as stated in Theorem* 5.1 *and, in addition, there exist a* $C^r \times C^{r,1}$ *(respectively* $C^{r-1} \times C^r$ *if* $r \geqq 1$*) global stable foliation* $\{\mathcal{F}^s(p) \in N_{\delta_0} \mid p \in W^{cs} \cap N_{\delta_0}\}$ *on* W^{cs} *and a* $C^r \times C^{r,1}$ *(respectively* $C^{r-1} \times C^r$ *if* $r \geqq 1$*) global unstable foliation* $\{\mathcal{F}^u(p) \in N_{\delta_0} \mid p \in W^{cu} \cap N_{\delta_0}\}$ *on* W^{cu}. *More specifically, there are functions* $\varphi^s: E^{cs}_{\delta_0} \times E^s_{\delta_0} \to E^{cu}_{\delta_0}$, $\varphi^u: E^{cu}_{\delta_0} \times E^u_{\delta_0} \to E^{cs}_{\delta_0}$ *of class* $C^r \times C^{r,1}$ *(respectively* $C^{r-1} \times C^r$ *if* $r \geqq 1$*) satisfying the properties* (i–iv) *for invariant foliations of section* 3 *such that* $\mathcal{F}^i(p) = \text{graph } \varphi^i(p, \cdot)$, $i = s, u$.

Theorem 5.1 was first explicitly stated in **[37]**, while Theorems 5.1, 5.2 with the tubular neighbourhoods replaced by the respective entire subspaces were essentially proved by Chow, Lin and Lu **[5]**. Their proofs can be easily adapted to our case with some minor modifications based on the following two observations. The regularity of the global manifolds and foliations at any point only depends on a neighbourhood of the positive orbit through that point for the centre-stable case or a neighbourhood of the negative orbit for the centre-unstable case. On the other hand, following their approach via the variation of constants formula, it is straightforward to verify that a constant $\delta_0 < \delta$ can be found so that all orbits starting in the smaller tubular neighbourhood N_{δ_0} stay in the larger one N_δ for all forward or backward time depending on the centre-stable or the centre-unstable case. For these reasons, we omit the proofs and refer to these two sources for the necessary modifications.

6. Extension lemma and proof of Theorem 3.1

In this section we use the following standard order notations: by $O(1)$, $O(\delta)$, $O(\delta^{-1})$ we mean that $\overline{\lim}_{\delta \to 0^+} O(1)$, $\overline{\lim}_{\delta \to 0^+} O(\delta)/\delta$, $\overline{\lim}_{\delta \to 0^+} O(\delta^{-1})\delta$ are constants, and by $o(1)$, $o(\delta)$ we mean $\overline{\lim}_{\delta \to 0^+} o(1) = 0$, $\overline{\lim}_{\delta \to 0^+} o(\delta)/\delta = 0$.

We now begin with the standard cut-off functions for invariant manifold theory. A cut-off function in one variable is a C^∞ function $\sigma: [0, \infty) \to [0, \infty)$ satisfying $\sigma(t) = 1$, $t \in [0, 1]$, $\sigma(t) = 0$, $t \geqq 2$ and $\sup_{0 \leqq t < \infty} (|\sigma(t)| + |\sigma'(t)| + |\sigma''(t)|) < \infty$. Let $\delta > 0$ and denote $\sigma_\delta(t) := \sigma(t/\delta)$. It is easy to verify that, in terms of our order notation,

$$\sigma_\delta = O(1), \quad \sigma'_\delta = O(\delta^{-1}), \quad \sigma''_\delta = O(\delta^{-2}).$$

Now cut-off functions in the Banach space $\mathbf{X}^\alpha$ are defined by

$$\rho_\delta(x_c) := \sigma_\delta(\|x_c\|), \quad \tilde{\rho}_\delta(x) := \sigma_\delta(\|x_c\|)\sigma_\delta(\|x_{su}\|_\alpha)$$

for $x_c \in E^c$, $x_{su} \in E^{su}$. The following properties will be used later: ρ_δ is a C^∞ function. Indeed, for $\|x_c\| \leqq \delta$ it is so since $\rho_\delta(\|x_c\|) = 1$ while for $\|x_c\| \geqq \delta$, $\|x_c\|$ is a C^∞ function since E^c is finite dimensional and σ_δ is C^∞ as well. Moreover, it satisfies the following estimates

$$\begin{aligned}
\rho_\delta(x_c) &= O(1),\\
\|D\rho_\delta(x_c)\| &\leqq |\sigma_\delta'(\|x_c\|)| \sup_{\|x_c\|\geqq\delta} \|D\| x_c \| \|\\
&= O(\delta^{-1}),\\
\|D^2\rho_\delta(x_c)\| &\leqq |\sigma_\delta''(\|x_c\|)|\Big(\sup_{\|x_c\|\geqq\delta} \|D \|x_c\| \|\Big)^2 + |\sigma_\delta'(\|x_c\|)| \sup_{\|x_c\|\geqq\delta} \|D^2 \|x_c\| \|\\
&= O(\delta^{-2}) + O(\delta^{-1})O(\delta^{-1})\\
&= O(\delta^{-2}),
\end{aligned} \tag{6.1a}$$

because $\|D \|x_c\| \| = O(1)$, $\|D^2 \|x_c\| \| = O(\|x_c\|^{-1})$ for $\|x_c\| > 0$. $\tilde{\rho}_\delta(x)$ is of class $C^{0,1}$ because the norm functions are $C^{0,1}$; it is C^∞ in N_δ since $\tilde{\rho}_\delta(x) = \rho_\delta(x_c)$ as $\sigma_\delta(\|x_{su}|_\alpha) = 1$ for $x \in N_\delta$. Moreover,

$$\begin{aligned}
&\tilde{\rho}_\delta(x) = 1, \quad x \in Q_\delta \quad \text{and} \quad \tilde{\rho}_\delta(x) = 0, \quad x \in \mathbf{X}^\alpha - Q_{2\delta}\ ,\\
&\operatorname{Lip} \tilde{\rho}_\delta \leqq \operatorname{Lip} \sigma_\delta \operatorname{Lip}(\|\cdot\|) \sup \sigma_\delta + \sup \sigma_\delta \operatorname{Lip} \sigma_\delta \operatorname{Lip}(\|\cdot\|_\alpha)\\
&\qquad = O(\delta^{-1}),
\end{aligned}$$

where $Q_\delta := \{x \in \mathbf{X}^\alpha \mid \|x_s\|_\alpha < \delta, \|x_c\| < \delta, \|x_u\| < \delta\}$, a box neighbourhood of the origin.

We remark that following Vanderbauwhede and Iooss [**37**] one obtains existence and smoothness of local centre manifolds for the locally defined equations (2.1) by applying the global centre manifold result to a globally extended equation of the form $\dot{x} = Ax + \tilde{\rho}_\delta(x)f(x)$. By the properties of $\tilde{\rho}_\delta$ discussed above, the function $F := \tilde{\rho}_\delta f$ satisfies the conditions of Theorem 5.1, and the desired result follows from the fact that the solutions of equations (2.1), (5.1) coincide in the neighbourhood Q_δ. Similarly, existence and regularity of the local centre-stable and centre-unstable manifolds together with the local stable and unstable foliations for equation (2.1) now follow directly from Theorems 5.1, 5.2 for the extended equation. Conversely, Sijbrand [**35**] has shown that for systems of ordinary differential equations every local centre manifold can be constructed in this way. We extend his idea to the infinite-dimensional case in the following two lemmas:

LEMMA 6.1. *Let* $W = \operatorname{graph} h$ *and* $h: U \subset E^c \to E^{su} \cap \mathbf{X}^\alpha$ *with* $0 \leqq \alpha < 1$ *be a* $C^{r+1,1}$, $r \geqq 0$ *function and U an open set in* E^c. *Then W is invariant for equation* (5.1) *if and only if h maps U into* $\mathbf{X}^1$, *the domain* $D(A)$ *of A, and the identity*

$$A_{su}h(x_c) + F_{su}(x_c + h(x_c)) = Dh(x_c)[A_c x_c + F_c(x_c + h(x_c))] \tag{6.2}$$

holds for all $x_c \in U$.

Proof. Suppose W is invariant. For every $x \in W$ with $x_c \in U$, let $\tilde{x}(t)$ be a solution curve in W containing x as an interior point, say $\tilde{x}(t_0) = x$ for some t_0. Then, the invariance of W implies $\tilde{x}_{su}(t) = h(\tilde{x}_c(t))$. Because $\tilde{x}(t)$ is a solution, then $h(x_c) = h(\tilde{x}_c(t_0)) = \tilde{x}_{su}(t_0) \in D(A)$ by definition. Differentiating the identity $\tilde{x}_{su}(t) = h(\tilde{x}_c(t))$ at $t = t_0$ yields (6.2).

Conversely, suppose h maps U into $D(A)$ and the identity (6.2) holds for all $x_c \in U$. Then, for every $x \in W$, let $\tilde{x}_c(t)$ be the solution of the ordinary differential equation $\dot{x}_c = A_c x_c + F_c(x_c + h(x_c))$ such that $\tilde{x}_c(t) \in U$ for small $|t|$ and $\tilde{x}_c(0) = x_c$. It can be verified directly that $\tilde{x}(t) := \tilde{x}_c(t) + h(\tilde{x}_c(t)) \in W$ is a solution curve in W containing x as an interior point. Indeed, by the definition for $\tilde{x}_{su}$ and identity (6.2), we have

$$\begin{aligned}\dot{\tilde{x}}_{su}(t) &= Dh(\tilde{x}_c(t))[A_c\tilde{x}_c(t) + F_c(\tilde{x}_c(t) + h(\tilde{x}_c(t)))] \\ &= A_{su}h(\tilde{x}_c(t)) + F_{su}(\tilde{x}_c(t) + h(\tilde{x}_c(t)))\end{aligned}$$

which together with the equation for $x_c(t)$ shows that the full equation is satisfied. □

Note that since the neighbourhood U in the proof above is arbitrary, the result is true regardless of whether the equation is locally or globally defined. In particular, it applies for equation (2.1). This observation will be used later.

LEMMA 6.2 (Extension lemma). *Assume hypotheses* (2.1*a*, *b*) *for equation* (2.1). *For an arbitrary $C^{r+1,1}$ local centre manifold $W^c_{loc} \subset \mathbf{X}^\alpha$, there are a small $\delta > 0$, a function $F \in C^{0,1}(\mathbf{X}^\alpha, \mathbf{X}) \cap C^{r,1}(N_\delta, \mathbf{X})$ and a global $C^{r,1}$ centre manifold $W^c \subset N_\delta$ of the new equation*

$$\dot{x} = Ax + F(x) \tag{6.3}$$

satisfying that $\sup_{x \in \mathbf{X}^\alpha} \|F(x)\| = o(\delta)$, $\mathrm{Lip}\, F = o(1)$, *as* $\delta \to 0^+$, *and that* $F|_{Q_\delta} = f|_{Q_\delta}$, $W^c \cap Q_\delta = W^c_{loc} \cap Q_\delta$.

Proof. By definition, $W^c_{loc} = \operatorname{graph} h$ for some small neighbourhood $V \subset \mathbf{X}^\alpha$ of the origin and a $C^{r+1,1}$ function $h\colon E^c \cap V \to E^{su} \cap \mathbf{X}^\alpha$ with $h(0) = 0$, $Dh(0) = 0$. Let ρ_δ and $\tilde{\rho}_\delta$ be cut-off functions as in (6.1). For $\delta > 0$ so small that $Q_{2\delta} \subset V$, define

$$\begin{aligned}h^c(x_c) &:= \rho_\delta(x_c)h(x_c) \\ W^c &:= \operatorname{graph} h^c \\ F(x) &:= \tilde{\rho}_\delta(x)f(x) + G(x_c)\end{aligned}$$

where $G\colon E^c \to E^{su}$ is defined by

$$\begin{aligned}G(x_c) := {} & Dh^c(x_c)[A_c x_c + \tilde{\rho}_\delta(x_c + h^c(x_c))f_c(x_c + h^c(x_c))] \\ & - \tilde{\rho}_\delta(x_c + h^c(x_c))f_{su}(x_c + h^c(x_c)) \\ & - \rho_\delta(x_c)\{Dh(x_c)[A_c x_c + f_c(x_c + h(x_c))] - f_{su}(x_c + h(x_c))\}.\end{aligned}$$

We claim that F, W^c have the desired properties.

First, we verify the extension properties for F and W^c. Indeed, $W^c \cap Q_\delta = W^c_{loc} \cap Q_\delta$ by construction. Moreover, $\rho_\delta|_{Q_\delta} = \tilde{\rho}_\delta|_{Q_\delta} = 1$, $G|_{Q_\delta} = 0$, and hence,

$F|_{Q_\delta} = f|_{Q_\delta}$. Next, we use Lemma 6.1 to demonstrate the invariance of the manifold W^c. Note that because the local centre manifold W^c_{loc} is invariant for equation (2.1), we have by Lemma 6.1.

$$A_{su}h(x_c) = Dh(x_c)[A_cx_c + f_c(x_c + h(x_c))] \\ - f_{su}(x_c + h(x_c)) := g(x_c), \quad \text{for} \quad x_c + h(x_c) \in V.$$

Moreover, since $\rho_\delta(x_c)$ is a scalar constant with respect to the operator A_{su}, $A_{su}h^c(x_c) = A_{su}(\rho_\delta(x_c)h(x_c)) = \rho_\delta(x_c)A_{su}h(x_c) = \rho_\delta(x_c)g(x_c)$. Thus, by definition of F, we have

$$\begin{aligned} A_{su}h^c(x_c) + F_{su}(x_c + h^c(x_c)) &= A_{su}h^c(x_c) + \tilde{\rho}_\delta(x_c + h^c(x_c))f_{su}(x_c + h^c(x_c)) \\ &\quad + Dh^c(x_c)[A_cx_c + \tilde{\rho}_\delta(x_c + h^c(x_c))f_c(x_c + h^c(x_c))] \\ &\quad - \tilde{\rho}_\delta(x_c + h^c(x_c))f_{su}(x_c + h^c(x_c)) - \rho_\delta(x_c)g(x_c) \\ &= Dh^c(x_c)[A_cx_c + F_c(x_c + h^c(x_c))]. \end{aligned}$$

This implies that W^c is invariant for equation (6.3) by Lemma 6.1. Furthermore, W^c is the centre manifold of the globally defined equation because $W^c - Q_{2\delta} = E^c - Q_{2\delta}$ on which the growth condition (5.2) characterising the global centre manifold is satisfied.

Finally, to prove $W^c \subset N_\delta$ and the estimates for F, we first collect some order estimates:

$$\|h(x_c)\|_\alpha = o(\|x_c\|), \ \|Dh(x_c)\|_\alpha = o(1), \text{ as } \|x_c\| \to 0^+, \text{ and } \operatorname{Lip} h|_{Q_{2\delta}} = O(1), \tag{6.4a}$$

because $h(0) = 0$, $Dh(0) = 0$ and h is of class $C^{r+1,1}$. Also,

$$\|g(x_c)\|_\alpha = o(\|x_c\|), \quad \text{as} \quad \|x_c\| \to 0^+, \quad \text{and} \quad \operatorname{Lip}(g|_{Q_{2\delta}}) = o(1), \\ \operatorname{Lip}(\rho_\delta g) = o(1), \quad \text{as} \quad \delta \to 0^+. \tag{6.4b}$$

Here, the first two estimates are true because of the definition of g and the preceding estimates for h together with the assumption that $f(0) = 0$, $Df(0) = 0$ and $f \in C^{r+1,1}$. The third estimate is true because

$$\operatorname{Lip}(\rho_\delta g) \leqq \sup_{\|x_c\| \leqq 2\delta} [\|D\rho_\delta(x_c)\| \ \|g(x_c)\| + |\rho_\delta(x_c)| \operatorname{Lip}(g)|_{Q_{2\delta}})],$$

where the first term is of order $O(\delta^{-1})o(\delta)$ and the second $O(1)o(1)$ by (6.1a) and the preceding estimates for g. We also claim that the following estimates hold:

$$\|Dh^c(x_c)\|_\alpha = o(1), \quad \operatorname{Lip} Dh^c = o(1)O(\delta^{-1}), \quad \text{as} \quad \delta \to 0^+. \tag{6.5}$$

We now estimate F and W^c by assuming this claim and then prove the claim.

To simplify the notation further, we denote

$$H(x_c) := A_cx_c + \tilde{\rho}_\delta(x_c + h^c(x_c))f_c(x_c + h^c(x_c)), \quad \theta(x_c) := x_c + h^c(x_c).$$

Thus, $G = Dh^cH - (\tilde{\rho}f_{su}) \circ \theta - \rho_\delta g$.

It is easy to see that $\|h^c(x_c)\|_\alpha = |\rho_\delta(x_c)| \ \|h(x_c)\|_\alpha \leqq O(1) \sup \|h|_{Q_{2\delta}}\|_\alpha = o(\delta)$ by (6.4a). Therefore, $W^c \subset N_\delta$ for sufficiently small δ. Also, $\|\theta(x_c)\|_\alpha = O(\|x_c\|)$.

The fact that $\|F(x)\| = o(\delta)$ is demonstrated as follows. Starting with the first term in the definition of F, we have $\|\tilde{\rho}_\delta f(x)\| = O(1)\,\|f|_{Q_{2\delta}}(x)\| = o(\|x\|_\alpha)$ by the hypothesis for the nonlinear term f. For the second term in the definition of F, we note that the foregoing estimate also implies $\|H(x_c)\| = O(\|x_c\|)$. Since

$$\|G(x_c)\| \leqq \|Dh^c(x_c)\|\;\|H|_{Q_{2\delta}}(x_c)\| + \|(\tilde{\rho}_\delta f_{su}) \circ \theta(x_c)\| + \|(\rho_\delta g)(x_c)\|,$$

we also have $\|G(x_c)\| = o(\delta)$ by (6.4b) and (6.5). Therefore, the estimate $\|F(x)\| = o(\delta)$ follows.

To show $\text{Lip}\, F = o(1)$, we also begin with the first term of F, and for which we have

$$\text{Lip}\,(\tilde{\rho}_\delta f) \leqq \text{Lip}\,\tilde{\rho}_\delta \sup \|f|_{Q_{2\delta}}\| + \sup |\tilde{\rho}_\delta|\;\text{Lip}\,(f|_{Q_{2\delta}}) = o(1)$$

because the first term after the inequality sign is of order $O(\delta^{-1})o(\delta)$ and the second $o(1)$ by (6.1b) together with the assumption for f. For the second term of F, we note that the foregoing estimate also implies

$$\text{Lip}\,[(\tilde{\rho}_\delta f) \circ \theta] \leqq \text{Lip}\,(\tilde{\rho}_\delta f)\;\text{Lip}\,\theta = o(1)$$

since $\text{Lip}\,\theta \leqq (1 + \sup \|Dh^c\|_\alpha) = O(1)$ by the claim. Hence, $\text{Lip}\, H = O(1) + o(1) = O(1)$ and $\text{Lip}\,(DhH) = o(1)$ because

$$\text{Lip}\,(DhH) \leqq \text{Lip}\,(Dh^c) \sup \|H|_{Q_{2\delta}}\| + \sup \|Dh^c\|_\alpha\;\text{Lip}\, H$$

for which the first term is of order $o(1)O(\delta^{-1})O(\delta)$ and the second $o(1)O(1)$ by the claim (6.5). We conclude from above and (6.4b) that

$$\text{Lip}\, G \leqq \text{Lip}\,(DhH) + \text{Lip}\,[(\tilde{\rho}_\delta f_{su}) \circ \theta] + \text{Lip}\,(\rho_\delta g) = o(1)$$

and, hence, $\text{Lip}\, F \leqq \text{Lip}\,(\tilde{\rho}_\delta f) + \text{Lip}\, G = o(1)$ as desired.

Finally, we complete the proof by proving claim (6.5). First, we have $\|Dh^c(x_c)\|_\alpha = o(1)$ because

$$\|Dh^c(x_c)\|_\alpha \leqq \|D\rho_\delta(x_c)\|\;\|h|_{Q_{2\delta}}(x_c)\|_\alpha + |\rho_\delta(x_c)|\;\|Dh|_{Q_{2\delta}}(x_c)\|_\alpha,$$

of which the first term is of order $O(\delta^{-1})o(\delta)$, or $O(1)o(1)$, and the second term $O(1)o(1)$. Next, to show $\text{Lip}\,(Dh^c) = o(1)O(\delta^{-1})$ we have

$$\text{Lip}\,(Dh^c) \leqq \max\,\{\text{Lip}\,(Dh|_{Q_\delta}),\ \text{Lip}\,(Dh^c|_{Q_{2\delta}-Q_\delta})\}$$

of which the first element is at most $O(1)$ since h is of class $C^{r+1,1}$ and the second element can be estimated

$$\begin{aligned}\text{Lip}\,(Dh^c|_{Q_{2\delta}-Q_\delta}) \leqq &\sup_{x_c \in Q_{2\delta}-Q_\delta} \|D^2\rho_\delta(x_c)\| \sup_{x_c \in Q_{2\delta}} \|h(x_c)\|_\alpha \\ &+ 2 \sup_{x_c \in Q_{4\delta}-Q_\delta} \|D\rho_\delta(x_c)\| \sup_{x_c \in Q_{2\delta}} \|Dh(x_c)\|_\alpha + \text{Lip}\,(Dh|_{Q_{2\delta}}),\end{aligned}$$

which is of order $o(1)O(\delta^{-1})$ because the first term is $O(\delta^{-2})o(\delta)$, or $O(\delta^{-1})o(1)$,
the second $O(\delta^{-1})o(1)$ and the third $O(1)$ by (6.1a) and (6.4a), respectively. □

A different version of this lemma can be obtained based on the following observations. Note that if $\text{Lip}\, f$ is not replaced by $o(1)$ in all the estimates above, then the estimates for the extended nonlinear term read

$\|F(x)\| = o(\delta) + O(\delta)\,\mathrm{Lip} f$ and $\mathrm{Lip}\, F = o(1) + \mathrm{Lip} f$. Moreover, if f is of class $C^{r,1}$ while W^c_{loc} belongs to the same $C^{r+1,1}$ class of manifolds as in the lemma, then the same result holds with these modified estimates of F. This remark will be used in Section 7 below.

We are now ready to give a proof for Theorem 3.1.

Proof of Theorem 3.1. This is simply an application of Lemma 6.2 and Theorem 5.2. Indeed, because of $\sup_{x \in \mathbf{X}^\alpha} \|F(x)\| = o(\delta)$, $\mathrm{Lip}\, F = o(1)$ by Lemma 6.2, the conditions that $\sup_{x \in \mathbf{X}^\alpha} \|\pi_{su} F(x)\| = o(\delta) < m = O(\delta)$ and $\mathrm{Lip}\, F = o(1) < \epsilon(r)$ for Theorem 5.2 are satisfied for sufficiently small δ. The smooth foliations for the locally defined equation (2.1) are now obtained by restricting the smooth foliations of the extended equation (5.1) to the neighbourhood $V := Q_{\delta_0} \subset Q_\delta$. □

We end this section with a remark that was used for the proof of Lemma 4.2. It is easy to see that for a given solution of equation (5.1) that starts on the global centre-stable manifold W^{cs} constructed from Theorem 5.1, all the backward extensions must stay on the manifold by the characterisation (5.2). Therefore, the same statement is true for any local centre-stable manifold constructed by the Extension Lemma 6.2 and Theorem 5.1.

7. Final remarks

(a) From the proof of the theorem and the remark after the Extension Lemma, it is easy to see that the following theorem is true:

THEOREM 7.1. *Assume the hypothesis* (2.1*a*) *for equation* (2.1) *and that the nonlinear term f is of class $C^{r,1}$ with $f(0) = 0$ and* Lip *f sufficiently small. Then the equations when restricted to two $C^{r+1,1}$ local centre manifolds of the origin are C^r conjugate.*

(b) In practice, there may be no need for the Extension Lemma because all centre manifolds in applications are constructed in the way described in Section 6. These manifolds are as smooth as the equation and the foliations only loose the Lipschitz continuity of the top partial derivative with respect to the base point. Hence, the conjugating map in this situation loses the top Lipschitz continuity as well. We have proved the following result:

THEOREM 7.2. *Assume the hypothesis* (2.1*a*) *for equation* (2.1) *and that the nonlinear term f is of class $C^{r,1}$ with $f(0) = 0$ and* Lip *f sufficiently small. Then the equations when restricted to two local centre manifolds of the origin that are constructed by the standard method are C^r conjugate.*

(c) For other types of infinite dimensional systems, e.g. the elliptic and hyperbolic equations studied by Vanderbauwhede and Iooss [**37**], we believe that invariant foliation theory can also be extended, and so can Theorems 2.1, 7.1, 7.2. The same result should also be expected for centre manifolds of normally hyperbolic invariant sets of diffeomorphisms and normally hyperbolic invariant manifolds of flows, for which a theory of invariant manifolds and foliations has long been established, cf., e.g., [**21, 12–14**]. The latter include the case of slow manifolds in the theory of singular perturbations.

(d) Returning to the conjugacy problem for inertial manifolds of an appropriate dissipative evolution equation discussed in the Introduction, we point out that the same result with C^0 conjugating map applies for $C^{0,1}$ inertial manifolds and $C^{0,1}$ equations. Furthermore, for $C^{1,1}$ equations, the C^0 regularity can be improved to C^1 when the spectral gap can be cut sufficiently away from the imaginary axis. Indeed, unlike the general case considered in the proof of Theorem 2.1, two inertial manifolds can be essentially regarded as lying in a common manifold, the entire phase space, that possesses a stable foliation. Thus, a proof for the C^0 or C^1 conjugacy statement follows directly from the invariant foliation theory proved by Chow, Lin and Lu [**5**] and the argument of Lemma 4.1. We remark that in general inertial manifolds are of class C^1 at best for $C^{r,1}$, $r \geqq 1$ systems under the conditions mentioned above. In fact, an example of an analytic equation having a C^1 inertial manifold which is not C^2 was given by Chow, Lu and Sell [**8**]. Note that even though this result applies only to inertial manifolds that are constructed by the standard method described in Section 1, it might be quite adequate for many practical purposes.

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