Links to the homework sets:
Homework 1
Homework 2
Homework 3

Homework 1

1. Consider the logistic map $x_{n+1} = f_\lambda(x_n) = 4\lambda x_n(1 - x_n)$. Outline the steps to find period-2 points for the map. Let $0 < \bar{y}_1 < \bar{y}_2$ be the period-2 points. Prove for $3/4 \le \lambda \le 1$ $\bar{y}_2(\lambda) \in [0, 1]$ and $\bar{y}_2$ is monotone increasing in λ.

2. Let f be a continuous map on the unit interval. Prove that if there is an interval $[a, b] \subseteq [0, 1]$ such that $f([a, b]) \supseteq [a, b]$, then there is a subinterval $I \subseteq [a, b]$ so that $f(I) = [a, b]$.

Solution: Because f is continuous and $f([a, b]) \supseteq [a, b]$, by the intermediate value theorem there must be points $x_1 \ne x_2$ in $[a, b]$ so that $f(x_1) = a$ and $f(x_2) = b$. WLOG assume $a \le x_1 < x_2 \le b$. Define a set by $A = \{x \le x_2 : f(x) \le a\}$. Then $A \ne \varnothing$ because $x_1 \in A$. Let $c = \sup A$. Then c exists since A is bounded above by x_2. Since either c is in A or a limit point of A, we must have $f(c) = a$ by continuity because the inequality $f(c) < a$ would imply a larger upper bound of A than c. Also $c < x_2$ since $f(x_2) = b > f(c) = a$. Next, define another set by $B = \{x \ge c : f(x) \ge b\}$. Then B is nonempty for containing x_2. Since it is bounded below by c, it has the greatest lower bound $d = \inf B$. By continuity again we must have $f(d) = b$. Also $d > c$ since $f(c) = a < b = f(d)$. Then the interval $J = [c, d]$ is a required interval, because by the definitions of c, d, the value of every point $x \in J$ must be $a \le f(x) \le b$. Otherwise, if $f(x) < a$, then $x \le d \le x_2$ and $c < x$ cannot be the supreme of A, or if $f(x) > b = f(d)$ and $x \ge c$, then d cannot be the infimum of B since $x \in B$ and $x < d$. This proves the result. $\square$

3. Let $0 \le a < b < c \le 1$ and f be a continuous map on the unit interval. Prove that if $f(a) = c,\ f(c) = b,\ f(b) = a$, then f has a period-k point for every $k \ge 1$.

Solution: Make this transformation $\bar{x} = 1 - x$ to the map to get $\bar{y} = 1 - f(1 - \bar{x}) := \bar{f}(\bar{x})$. Then the points $a < b < c$ are transformed to $\bar{a} < \bar{b} < \bar{c}$ with $\bar{a} = 1 - c,\ \bar{b} = 1 - b,\ \bar{c} = 1 - a$. So for the new map $\bar{f}$ $\{\bar{a}, \bar{b}, \bar{c}\}$ is the period-3 orbit in the order the same result is proved in the lecture. $\square$

Homework 2

1. Let T be the tent map on $[0, 1]$, and let $\sigma(x_0)$ and $I_{s_0 \ldots s_n}$ be defined as in lecture. Prove the following:

1. $\sigma(x_0) = s_0 s_1 \ldots s_n \ldots \Rightarrow \sigma(f^k(x_0)) = s_k s_{k+1} \ldots s_n \ldots$.
2. $I_{s_0 \ldots s_n} = I_{s_0 \ldots s_{n-1}} \cap (f^n)^{-1}(I_{s_n})$.
3. $I_{s_0 \ldots s_n} = I_{s_0 \ldots s_n L} \cup I_{s_0 \ldots s_n R}$ and $|I_{s_0 \ldots s_n}| = (1/2)^{n+1}$.
4. $I_{s_0 \ldots s_n} \cap I_{s'_0 \ldots s'_n} \ne \varnothing$ iff $s_k = s'_k$ for all $k = 0, 1, \ldots, n$.
5. f^{n+1} is monotone increasing on $I_{s_0 \ldots s_n} \Leftrightarrow n_{s_0 \ldots s_n} =$ even, where $n_{s_0 \ldots s_n}$ is the number of Rs in $\{s_0, \ldots, s_n\}$.

Solution: 1. By definition $\sigma(x_0) = s_0 s_1 \ldots s_n \ldots$ if and only if $f^n(x_0) \in I_{s_n}$ for $n = 0, 1, \ldots$. For $x'_0 = f^k(x_0)$, since $f^n(x'_0) = f^n(f^k(x_0)) = f^{n+k}(x_0) \in I_{s_{n+k}} = I_{s'_n}$, we have $s'_n = s_{n+k}$ for $n = 0, 1, \ldots$. So by definition $\sigma(x'_0) = s'_0 s'_1 \ldots s'_n \cdots = s_k s_{k+1} \ldots$.

2. Recall by definition $I_{s_0 \ldots s_n} = \{x : f^k \in I_{s_k},\ k = 0, 1, \ldots, n.\}$. So for every $x \in I_{s_0 \ldots s_n}$, $x \in I_{s_0 \ldots s_{n-1}}$ since $f^k(x) \in I_{s_k}$ for every $k = 0, 1, \ldots, n-1$, and $f^n \in I_{s_n}$, implying $x \in I_{s_0 \ldots s_{n-1}} \cap (f^n)^{-1}(I_{s_n})$, and $I_{s_0 \ldots s_n} \subset I_{s_0 \ldots s_{n-1}} \cap (f^n)^{-1}(I_{s_n})$. Conversely, for every $x \in I_{s_0 \ldots s_{n-1}} \cap (f^n)^{-1}(I_{s_n})$, $f^k(x) \in I_{s_k}$ for every $k = 0, 1, \ldots, n-1$ since $x \in I_{s_0 \ldots s_{n-1}}$ and $f^n(x_0) \in I_{s_n}$. So by definition, $x \in I_{s_0 \ldots s_n}$, implying $I_{s_0 \ldots s_{n-1}} \cap (f^n)^{-1}(I_{s_n}) \subset I_{s_0 \ldots s_n}$. This two-ways inclusion argument proves the identity.

3. For the first part, it follows by definition for $I_{s_0 \ldots s_n}$ that $x \in I_{s_0 \ldots s_n}$ if and only if $f^k(x_0) \in I_{s_k}$ for $k = 0, 1, \ldots, n$. Since $I = I_L \cup I_R$ and $I_L \cap I_R = \varnothing$, $f^{n+1}(x_0)$ is in either I_L or I_R. So by definition, $I_{s_0 \ldots s_n} \subset I_{s_0 \ldots s_n L} \cup I_{s_0 \ldots s_n R}$. Since the reverse inclusion is trivial, we have the equality as a result.

Next, for the second part, we first note a general result for any affine linear function f on $\mathbb{R}$ with slope $r \ne 0$. It is always true that the image of an interval is an interval and the pre-image of an interval is also an interval. More importantly, the interval lengths are scaled exactly by $|r|$ or $1/|r|$ accordingly. That is, if $J = f(I)$ for intervals I, J, $|f(I)| = |r||I|$ and $|f^{-1}(J)| = |J|/|r|$. Now for the tent map $f = T$ on $[0,1]$, f is linear on $I_L = [0,1/2)$ and $I_R = [1/2,1]$ with $r = 2$ and $r = -2$ respectively. So $|r| = 2$ on I_L and I_R. Now for the second part, we can prove first by a similar argument to (2) above that $I_{s_0\ldots s_n} = I_{s_0} \cap f^{-1}(I_{s_1\ldots s_n})$. Since f is linear on I_{s_0} and $I_{s_1\ldots s_n} \subset I$ is in the range of $f|_{I_{s_0}}$, we have $I_{s_0\ldots s_n} = I_{s_0} \cap f^{-1}(I_{s_1\ldots s_n}) = (f|_{I_{s_0}})^{-1}(I_{s_1\ldots s_n})$. By the result above we have $|I_{s_0\ldots s_n}| = |I_{s_1\ldots s_n}|/2$. So by a recursive argument, we have $|I_{s_0\ldots s_n}| = |I_{s_2\ldots s_n}|/2^2 = \cdots = 1/2^{n+1}$ because $|I_L| = |I_R| = 1/2$ for $n = 0$.

4. By definition, $x \in I_{s_0\ldots s_n} \cap I_{s'_0\ldots s'_n} \neq \varnothing$ iff $f^k(x) \in I_{s_k} \cap I_{s'_k} \neq \varnothing$ for all $k = 0,1,\ldots,n$. Since $I_L \cap I_R = \varnothing$, we must have $s_k = s'_k$ for all $k = 0,1,\ldots,n$.

5. We will prove a more general statement that f^{n+1} is a linear map on $I_{s_0\ldots s_n}$ and that it is monotone increasing on $I_{s_0\ldots s_n} \Leftrightarrow r_{s_0\ldots s_n} =$ even, where $r_{s_0\ldots s_n}$ is the number of Rs in $\{s_0,\ldots,s_n\}$, referred to as the R-number of sequence $s_0,\ldots,s_n$. By induction, this statement is true for $n = 0$ for which f is linear on I_L and I_R and is increasing on I_L and decreasing on I_R. Assume the statement is true for $n-1$ for all n-length sequence $s_0 \ldots s_{n-1}$. Now consider the case for n. Because of (2), $I_{s_0\ldots s_n} = I_{s_0\ldots s_{n-1}} \cap (f^n)^{-1}(I_{s_n})$. By induction hypothesis, f^n is linear on $I_{s_0\ldots s_{n-1}}$ and is increasing iff $r_{s_0\ldots s_{n-1}}$ is even. Since the image of f^n on $I_{s_0\ldots s_n}$ is in I_{s_n} and f is linear on I_{s_n}, the composition $f^{n+1} = f \circ f^n$ is also linear because the composition of linear functions is a linear function. Last, if $s_n = L$ then f keeps the same inclination of f^n and $r_{s_0\ldots s_{n-1}} =$ even iff $r_{s_0\ldots s_n} =$ even. If $s_n = R$ then f reverses the inclination of f^n and $r_{s_0\ldots s_{n-1}} =$ even iff $r_{s_0\ldots s_n} =$ odd. This proves (5). $\square$

2. Let $B = \{0,1\}$ be the binary symbol set. Define $\delta(s,t) = 0$ if $s = t$ and $\delta(s,t) = 1$ if $s \neq t$ for $s,t \in B$. Show that δ is a metric on B.

3. Let $S_2 = \{s = s_0 s_1 \ldots : s_i \in B\}$ be the set of all binary sequences. Define

$$d(s,s') = \sum_{i=0}^{\infty} \frac{\delta(s_i,s'_i)}{2^{i+1}}, \text{ for } s,\ s' \in S_2.$$

Show that d is a metric in S_2.

Back to Top

Homework 3

1. Let $T(x)$ be the tent map with slope 3 from the unit interval $I = [0,1]$ to $\mathbb{R}$: $T(x) = 3x, 0 \le x < 1/2$ and $T(x) = 3(1-x), 0 \le x < 1/2$. Let $C = \{x \in I : f^n(x) \in I, \text{ for all } n = 0,1,2,\ldots\}$.

a. Prove that C is the Cantor mid-third set, i.e., it is closed, uncountable, containing no intervals, and every point of C is a limit point of C.

b. Prove that the dynamics (T, C) is topologically conjugate to the shift dynamics (σ, S_2) on 2-symbols.

c. Let $C^c = I \setminus C$, the complement of C. Show that C^c is the sum of countable disjoint intervals and the total length is $|C^c| = 1$.

Solution: Let $C_n = \{x : f^i(x) \in I, 0 \le i \le n\}$. Then $C_0 = I$, $C_1 = I_L \cup I_R$ where $I_L = [0,1/3]$ and $I_R = [2/3,1]$, and C_1 is obtained by removing the mid-third open interval from C_0. By induction, we can prove the following properties:

(1) C_n is the union of 2^n many closed intervals.

(2) Each interval in C_n corresponds to an itinerary of length n, $s_0 s_1 \ldots s_{n-1}$, $s_i \in \{L, R\}$, and denote it by $I_{s_0 s_1 \ldots s_{n-1}}$.

(3) The itinerary is defined by $x \in I_{s_0 s_1 \ldots s_{n-1}}$ iff $f^i(x) \in I_{s_i}, 0 \le i \le n-1$.

(4) The length of $I_{s_0 s_1 \ldots s_{n-1}}$ is $1/3^n$.

(5) $I_{s_0 s_1 \ldots s_{n-1} L}$ and $I_{s_0 s_1 \ldots s_{n-1} R}$ are obtained by removing the mid-third open interval of $I_{s_0 s_1 \ldots s_{n-1}}$.

(6) Let $r(s)$ be the number of symbol Rs of any symbol sequence s. Then $r(s_0 s_1 \ldots s_{n-1}) =$ even iff $I_{s_0 s_1 \ldots s_{n-1} L}$ is the left third subinterval of $I_{s_0 s_1 \ldots s_{n-1}}$.

(7) f^n is a linear map defined on $I_{s_0 s_1 \ldots s_{n-1}}$ whose image is the interval I, and whose slope is 3^n if $r(s_0 s_1 \ldots s_{n-1}) =$ even and -3^n if $r(s_0 s_1 \ldots s_{n-1}) =$ odd.

(8) $C_n \subset C_{n-1}$ for $n \geq 1$, and the complement C_n relative to C_{n-1}, $C_{n-1} \setminus C_n$ is the union of 2^{n-1} open intervals, each is of length $1/3^n$ and is one of the mid-third interval removed from $I_{s_0 s_1 \ldots s_{n-2}}$ for some $s = s_0 s_1 \ldots s_{n-2}$.

(9) For any itinerary sequence $s_0 s_1 \ldots s_{n-2}$, f^n maps the mid-third interval of $I_{s_0 s_1 \ldots s_{n-2}}$ outside of I, hence, every point from the mid-third interval of $I_{s_0 s_1 \ldots s_{n-2}}$ escapes I, thus not in C.

As a result, $C = \cap_{n=0}^{\infty} C_n$. It is nonempty and closed because $\{C_n\}$ is a set of nested and closed subsets. It is uncountable because the itinerary map $\phi : C \to S_2$ with $\phi(x) = s_0 s_1 \ldots$ for each point $x \in C$ (with $f^i(x) \in I_{s_i},\ i \geq 0$) is 1-1 and onto, and S_2 is clearly uncountable. Specifically, $\{x\} = \cap_{n=0}^{\infty} I_{s_0 \ldots s_n}$ iff $\phi(x) = s_0 s_1 \ldots$. Here, we used the fact that because $I_{s_0 \ldots s_n} \subset I_{s_0 \ldots s_{n-1}}$ and $|I_{s_0 \ldots s_n}| \to 0$ as $n \to 0$, the intersection $\cap_{n=0}^{\infty} I_{s_0 \ldots s_n}$ contains a unique point. It contains no interval because for any $x \in C$ with itinerary $\phi(x) = s_0 s_1 \ldots$, and any interval U of x, there must be a sufficiently large n so that $x \in I_{s_0 \ldots s_{n-1}} \subset U$ because the length of $|I_{s_0 \ldots s_{n-1}}| = 1/3^n \to 0$, and the removed mid-third interval from $I_{s_0 \ldots s_{n-1}}$ is not in C. Last, for any point $x \in C$ and any ϵ-neighborhood, $B_\epsilon(x)$, there must be a large N so that for all $n \geq N$, $I_{s_0 \ldots s_n} \subset B_\epsilon(x)$ because $|I_{s_0 \ldots s_{n-1}}| = 1/3^n \to 0$, where $s = \phi(x)$. Let x' be a point whose itinerary $\phi(x') = s_0' s_1' \ldots$ is the same as x's except at the $(n+2)$nd position for which $s_{n+1}' \neq s_{n+1}$. By definition, $x' \in I_{s_0 \ldots s_n} \subset B_\epsilon(x)$ and $x \neq x'$. As a result x is a limit point of C.

(b) By the construction of C above, we have obtained the conjugacy map $\phi : C \to S_2$ by equipping S_2 with the usual symbolic sequence metric $d(s, s') = \sum_{n=0}^{\infty} \delta(s_i, s_i')/2^{i+1}$ and δ being the discrete metric. On can show that ϕ is 1-1, onto, continuous, and the inverse is also continuous.

(c) C^c is the union of all mid-third intervals removed from the construction of C. Its length is

$$|C^c| = \sum_{n=1}^{\infty} 2^{n-1} \frac{1}{3^n} = \frac{1}{3} \frac{1}{1 - 2/3} = 1$$

where 2^{n-1} is the number of mid-third intervals removed at step n, each is of length $1/3^n$. Therefore, $|C| = 1 - |C^c| = 0$ follows.

2. Let f be the piecewise function as below

$$f(x) = \begin{cases} 2x, & 0 \leq x < 1/2 \\ 3/4 - x, & 1/2 \leq x < 3/4 \\ x - 3/4, & 3/4 \leq x \leq 1 \end{cases}$$

Find the piecewise density function for the natural measure of f. Show work.

Solution Key: Transition matrix is P or $P^T = \begin{bmatrix} 1/2 & 0 & 1 & 1 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix}$. Here $p_{ij} = |f^{-1}(I_i) \cap I_j|/|I_j|$, with, for example,

$$p_{11} = |f^{-1}(I_1) \cap I_1|/|I_1| = (1/8)/(1/4)$$

with $I_1 = [0, 1/4), I_2 = [1/4, 1/2), I_3 = [1/2, 3/4), I_4 = [3/4, 1]$. Probability eigenvector of eigenvalue 1 is $Pw = w$, $w = [1/2, 1/4, 1/8, 1/8]^T$. And the piecewise density function is $d = (2, 1, 1/2, 1/2)$.