Homework of MATH 939

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Homework 1

1. Consider the logistic map $x_{n+1} = f_\lambda(x_n) = 4\lambda x_n(1 - x_n)$. Outline the steps to find period-2 points for the map. Let $0 < \bar{y}_1 < \bar{y}_2$ be the period-2 points. Prove for $3/4 \le \lambda \le 1$ $\bar{y}_2(\lambda) \in [0,1]$ and $\bar{y}_2$ is monotone increasing in λ.

2. Let f be a continuous map on the unit interval. Prove that if there is an interval $[a,b] \subseteq [0,1]$ such that $f([a,b]) \supseteq [a,b]$, then there is a subinterval $I \subseteq [a,b]$ so that $f(I) = [a,b]$.

3. Let $0 \le a < b < c \le 1$ and f be a continuous map on the unit interval. Prove that if $f(a) = c,\ f(c) = b,\ f(b) = a$, then f has a period-k point for every $k \ge 1$.

Homework 2

1. Let T be the tent map on $[0,1]$, and let $\sigma(x_0)$ and $I_{s_0\ldots s_n}$ be defined as in lecture. Prove the following:

1. $\sigma(x_0) = s_0 s_1 \ldots s_n \ldots \Rightarrow \sigma(f^k(x_0)) = s_k s_{k+1} \ldots s_n \ldots$.
2. $I_{s_0\ldots s_n} = I_{s_0\ldots s_{n-1}} \cap (f^n)^{-1}(I_{s_n})$.
3. $I_{s_0\ldots s_n} = I_{s_0\ldots s_n L} \cup I_{s_0\ldots s_n R}$ and $|I_{s_0\ldots s_n}| = (1/2)^{n+1}$.
4. $I_{s_0\ldots s_n} \cap I_{s'_0\ldots s'_n} \ne \varnothing$ iff $s_k = s'_k$ for all $k = 0, 1, \ldots, n$.
5. f^{n+1} is monotone increasing on $I_{s_0\ldots s_n} \Leftrightarrow n_{s_0\ldots s_n} =$ even, where $n_{s_0\ldots s_n}$ is the number of Rs in $\{s_0, \ldots, s_n\}$.

2. Let $B = \{0, 1\}$ be the binary symbol set. Define $\delta(s,t) = 0$ if $s = t$ and $\delta(s,t) = 1$ if $s \ne t$ for $s, t \in B$. Show that δ is a metric on B.

3. Let $S_2 = \{s = s_0 s_1 \ldots\ : s_i \in B\}$ be the set of all binary sequences. Define

$$d(s, s') = \sum_{i=0}^{\infty} \frac{\delta(s_i, s'_i)}{2^{i+1}}, \text{ for } s,\ s' \in S_2.$$

Show that d is a metric in S_2.

Homework 3

1. Let $T(x)$ be the tent map with slope 3 from the unit interval $I = [0,1]$ to $\mathbb{R}$: $T(x) = 3x, 0 \le x < 1/2$ and $T(x) = 3(1-x), 0 \le x < 1/2$. Let $C = \{x \in I : f^n(x) \in I,\ \text{for all } n = 0, 1, 2, \ldots\}$.

a. Prove that C is the Cantor mid-third set, i.e., it is closed, uncountable, containing no intervals, and every point of C is a limit point of C.

b. Prove that the dynamics (T, C) is topologically conjugate to the shift dynamics (ϕ, S_2) on 2-symbols.

c. Let $C^c = I \setminus C$, the complement of C. Show that C^c is the sum of countable disjoint intervals and the total length is $|C^c| = 1$.

2. Let f be the piecewise function as below

$$f(x) = \begin{cases} 2x, & 0 \leq x < 1/2 \\ 3/4 - x, & 1/2 \leq x < 3/4 \\ x - 3/4, & 3/4 \leq x \leq 1 \end{cases}$$

Find the piecewise density function for the natural measure of f. Show work.

Homework 4

1. Let $C^0(\mathbb{R}, \mathbb{R})$ be the set of all uniformly continuous and bounded functions in $\mathbb{R}$. and let $||f||_0 = \sup_{\mathbb{R}} |f(x)|$. Prove that $||f||_0$ defines a norm for C^0 and that C^0 with the norm is a Banach space.

2. Let $C^1(\mathbb{R}, \mathbb{R})$ be the set of all differentiable functions for which both the functions and their derivatives are uniformly continuous and bounded in $\mathbb{R}$. Let $||f||_1 = \sup_{\mathbb{R}}(|f(x)| + |f'(x)|)$. Prove that $||f||_1$ defines a norm for C^1 and that C^1 with the norm is a Banach space. (Hint: Use the fundamental theorem of calculus to show that f is differentiable with derivative $f' = h$ if and only if $f(x) = f(x_0) + \int_{x_0}^{x} h(t)dt$.)

3. Let A be an $n \times n$ invertible matrix and B be an $m \times m$ matrix. Let $g \in C^0(\mathbb{R}^n, \mathbb{R}^m) := X$ and define $T : X \to X$ by $T(g)(x) = Bg(A^{-1}x)$, i.e. $T(g) = B \circ g \circ A^{-1}$.

a. Prove that T is in $L(X, X)$, the Banach space of all bounded linear maps from X to X and $||T|| \leq ||B||$.

b. Prove that T is differentiable in g and that $[D_g T]h = B \circ h \circ A^{-1}$, a linear map from X to X.

c. Let $T^k = T \circ T \cdots \circ T$ be the kth iterate of T in composition. Prove that $T^k(g)(x) = B^k g(A^{-k}x)$.