M428 Exam 2 Solution Key

Name: ______________________ 8-Digits NUID: ________________

1 (30 pts) On-line orders for a product come in according to a Poisson process at a mean rate of 30 per hour. Two candidates are applying for the job to fill out the orders (before shipping). Both candidates have an exponential distribution for service time per order, with candidate X having a mean of 1.2 minutes and candidate Y having a mean of 1.5 minutes.

(a) Find the expected waiting time (in minutes) for an order in the system (before shipping) if X is hired.

(b) Find the same expected waiting time if Y is hired.

(c) If the management has decided that for each minute saved for an order the handler/server should make \$1 more per hour, how much more should handler X make per hour than handler Y should?

Solution Key: (a) Use M/M/1 model. $\lambda = 30/60 = 1/2$ per minute. $\mu_X = 1/1.2 = 5/6$ per minute. $\rho = \rho_X = \lambda/\mu_X = 3/5$.

$$W_X = \frac{1}{\lambda}\frac{\rho_X}{1-\rho_X} = 3 \text{ minutes}$$

(b) Similarly, with $\mu_Y = 1/1.5 = 2/3$ per minute we have

$$W_Y = \frac{1}{\lambda}\frac{\rho_Y}{1-\rho_Y} = 6 \text{ minutes}$$

(c) X should make $\$1 \times (W_Y - W_X) = \$3$ per hour more than Y would.

2 (35 pts) A gas station has 3 gas pumps. The amount of time that a pump works before breaking down has an exponential distribution with a mean of 3 weeks. It has only one mechanics qualified to repair broken pumps, who is able to fix 10 pumps per week and the repair time is also exponentially distributed.

(a) Let the state of the system be the number of pumps out of work, construct the rate diagram for this queueing system.

(b) Write down the balance equations.

(c) What is the fraction of time all pumps are working?

(d) What is the expected time a broken pump is not in service?

Solution Key: The maximal number of non-working pumps is 3. Each pump has an average working time of 3 weeks that is exponentially distributed, and thus has a $\lambda = 1/3$ per week input rate to the repairing queuing system. Because of all working times are exponentially distributed with the same λ, we have

$$\lambda_0 = 3\lambda = 1,\ \lambda_1 = 2\lambda = 2/3,\ \lambda_2 = \lambda = 1/3,\ \lambda_k = 0 \text{ for } k \geq 3.$$

Also, since there is only one mechanic ($s = 1$) we have

$$\mu_k = \mu = 10 \text{ per week for } k = 1, 2, 3.$$

(a)

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda_0 = 1"] & 1 \arrow[l, bend left, "\mu_1 = 10"] \arrow[r, bend left, "\lambda_1 = 2/3"] & 2 \arrow[l, bend left, "\mu_2 = 10"] \arrow[r, bend left, "\lambda_2 = 1/3"] & 3 \arrow[l, bend left, "\mu_3 = 10"] \end{tikzcd}$$

(b) Let $P = \{P_0, P_1, P_2, P_3\}$ be the probability distribution of the state space ($P_k = 0$ for $k \geq 4$). Then

$$\begin{array}{lrcl} \text{State} & \text{In-Rate} & = & \text{Out-Rate} \\ 0: & \mu_1 P_1 & = & \lambda_0 P_0 \\ 1: & \lambda_0 P_0 + \mu_2 P_2 & = & \mu_1 P_1 + \lambda_1 P_1 \\ 2: & \lambda_1 P_1 + \mu_3 P_3 & = & \mu_2 P_2 + \lambda_2 P_2 \\ 3: & \lambda_2 P_2 & = & \mu_3 P_3 \end{array}$$

(c)

$$C_0 = 1,\ C_1 = \frac{\lambda_0}{\mu_1} = 1/10,\ C_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2} = 1/150,\ C_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} = 1/4500$$

$$P_0 = 1/\textstyle\sum_{n=0}^{3} C_n = 0.9034, \quad P_n = C_n P_0,\ n = 0, 1, 2, 3$$

$$P = \{P_0, P_1, P_2, P_3\} = \{0.9034,\ 0.0903,\ 0.0060,\ 0.0002\}$$

The fraction of time that all pumps are working is $P_0 = 0.9034$, about 90% of the time.

(d) $L = 0P_0 + 1P_1 + 2P_2 + 2P_3 = 0.1030$. Since

$$\bar{\lambda} = \sum_{k=0}^{3} \lambda_k P_k = 0.9657$$

and a broken pump is expected to be out of service for or to be fixed in

$$W = L/\bar{\lambda} = 0.1067 \text{ week} \approx 0.75 \text{ day} \approx 18 \text{ hours}.$$

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3 (35 pts) Dave's Photography Store has a weekly inventory policy for one brand of camera: if the store has cameras on hand at the end of week, then Dave will do nothing, otherwise Dave will order 2 cameras for the start of next week. Assume the demand for this particular brand of camera has a Poisson distribution for a mean of one camera per week.

(a) Let D be the demand random variable during each week when the store is in business. Find

$$P(D = 0), \ P(D = 1), \ P(D \geq 2).$$

(b) Let $X(t)$ be the number of cameras in stock at the end of week t. Find the 1-step transition matrix. Choose any two entries of the matrix, say $p_{00}, \ p_{11}$, and explain how they are derived.

(c) Find the probability that the store has 2 cameras at the end of week 3 given that the store has 1 camera at the end of week 1.

(d) Find the expected first passage time from the store's having to order 2 cameras to ordering none.

Solution Key:

(a) By Poisson distribution $P(D(t) = n) = \frac{(\alpha t)^n \exp(-\alpha t)}{n!}$ with $\alpha = 1, \ t = 1$, we have

$$P(D = 0) = 0.3679, \ P(D = 1) = 0.3679, \ P(D \geq 2) = 1 - P(D = 0) - P(D = 1) = 0.2642.$$

(b) State space $S = \{0, 1, 2\}$. For p_{00}, $X(t) = 0$, Dave would order 2 cameras to start next week. Since $X(t + 1) = 0$, the demand $D(t + 1)$ is at least 2. So $p_{00} = P(D \geq 2) = 0.2642$. For p_{11}, $X(t) = 1$, No new cameras would be ordered. The new week starts with the carryover 1 camera. Since $X(t + 1) = 1$, no camera is sold and $D(t + 1) = 0$. So $p_{11} = P(D = 0) = 0.3679$. All other entries are derived similarly. As a result, the transition matrix is

$$P = \begin{bmatrix} 0.2642 & 0.3679 & 0.3679 \\ 0.6321 & 0.3679 & 0 \\ 0.2642 & 0.3679 & 0.3679 \end{bmatrix},$$

(c) Since

$$P^2 = \begin{bmatrix} 0.3996 & 0.3679 & 0.2325 \\ 0.3996 & 0.3679 & 0.2325 \\ 0.3996 & 0.3679 & 0.2325 \end{bmatrix},$$

we have $P(X(t + 2) = 2 | X(t) = 1) = 0.2325$, the $(1, 2)$-entry of P^2.

(d) To order 2 cameras is to have $X(t) = 0$. To order none is to have either $X(t) = 1$ or $X(t) = 2$. So we need to find both μ_{01} and μ_{02}.

To find μ_{01}, let $\vec{\mu}_1 = [\mu_{01}, \ \mu_{11}, \ \mu_{21}]^T$. Then we know

$$[I - Q(1)]\vec{\mu}_1 = \mathbf{1} = [1, 1, 1]^T$$

where $Q(1)$ is the matrix of P with the 1-column (i.e. column 2) replaced by the zero vector. Solve this system of equations by rref as below

$$\text{rref}([I - Q(1) \vdots \mathbf{1}]) = \begin{bmatrix} 1.0000 & 0 & 0 & 2.7181 \\ 0 & 1.0000 & 0 & 2.7181 \\ 0 & 0 & 1.0000 & 2.7181 \end{bmatrix}$$

showing $\mu_{01} = 2.7181$ weeks. Similarly, for μ_{02}, we solve $\vec{\mu}_2$ from $[I - Q(2)]\vec{\mu}_2 = \mathbf{1}$ to get

$$\text{rref}([I - Q(2) \vdots \mathbf{1}]) = \begin{bmatrix} 1.0000 & 0 & 0 & 4.3002 \\ 0 & 1.0000 & 0 & 5.8822 \\ 0 & 0 & 1.0000 & 4.3002 \end{bmatrix}$$

showing $\mu_{02} = 4.3002$ weeks.

Because $P^\infty \approx P^2$ for having identical rows, we have $\pi_1 = 0.3679$ and $\pi_2 = 0.2325$. So the expected first passage time from the store's having to order 2 cameras to ordering none is

$$\mu = \frac{\pi_1}{\pi_1 + \pi_2}\mu_{01} + \frac{\pi_2}{\pi_1 + \pi_2}\mu_{02} = 3.3308 \text{ weeks}$$

End of Exam 2