SS2021 M428 Exam 1 Solution Key

Print Your Name Legibly:______________________________ Score:__________

1(30 pts) Consider the linear programming problem

Maximize $z = 3x_1 + 2x_2$
subject to

$$\begin{aligned} x_1 \quad\;\; &\le 5 \\ x_1 + x_2 &\le 8 \\ x_1 \ge 0,\ x_2 &\ge 0. \end{aligned}$$

(a) Find the dual LP problem.

(b) Solve the dual LP problem graphically, including all details for credit.

(c) Without solving the primal LP problem, what is its optimal value z^* based on (a) and (b)?

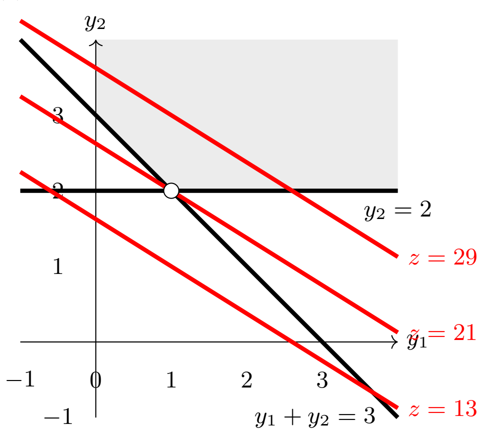
(d) If the right-hand side of the second constraint for the primal LP problem changes from 8 to 7.5, without solving the primal LP problem again what is the primal optimal value z^* expected to be? Explain why.

Solution: (a)

Minimize $z = 5y_1 + 8y_2$
subject to

$$\begin{aligned} y_1 + y_2 &\ge 3 \\ y_2 &\ge 2 \\ y_1 \ge 0,\ y_2 &\ge 0. \end{aligned}$$

(b)



Optimal solution to the dual: $(z^*, y_1^*, y_2^*) = (21, 1, 2)$.

(c) $z^* = 21$.

(d) $z^* = 21 + y_2^* \Delta b_2 = 21 + 2(-.5) = 20$, because y_2^* is the shadow price for the primal LP's second constraint.

2(35 pts) The augmented matrix form (tableau form) for a LP problem: maximize $z = c^T x$ subject to $Ax \le b,\ x \ge 0$ is given as follows:

		z	x_1	x_2	x_3	x_4	x_5	rhs
	z	1	-1	-2	-3	0	0	0
basic variable	x_4	0	1	-1	1	1	0	0
	x_5	0	2	1	1	0	1	4

(a) Is $x_1 = x_2 = x_3 = 2$ a feasible point of the problem? Justify your answer.

(b) Find the optimal solution of the problem by using the tableau simplex method only. To receive credit, you must show the feasible echelon form for each iteration of the method together with application of optimality test, ratio test, and elementary row operations, such as 2R1+R0 for example, from one echelon form to the next.

(c) Let $E = E_k \cdots E_1$ where $E_i, i = 1, \ldots, k$, are the elementary matrices corresponding to the elementary row operations used for the simplex method above. What is E?

(d) If the right-hand of the constraint is changed from $\begin{bmatrix}0\\4\end{bmatrix}$ to $\begin{bmatrix}1\\4\end{bmatrix}$, use (c) only to find the new optimal solution. Explain why your solution is the optimal solution. Any other method will not receive the full point.

Solution: (a) No, because the constraint #2 is not satisfied at the point: $2 * x_1 + x_2 + x_3 = 6 > 4$ rather than ≤ 4.

(b)

		z	x_1	x_2	x_3	x_4	x_5	rhs	ratio test
iteration 1	z	1	-1	-2	-3	0	0	0	
basic variable	x_4	0	1	-1	1	1	0	0	$0^+/1 = 0^+ < 4$
	x_5	0	2	1	1	0	1	4	$4/1 = 4 > 0^+$

Feasible solution: $(z, x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 0, 0, 4)$.
Optimality Test: No, since there are negative coefficients for the z-equation.
Entering variable: x_3 for having the most negative z-equation coefficient.
Leaving variable: x_4 for being the smallest 0^+ ratio.
Subsequent Row Operations: 3*R1+R0, -R1+R2.

		z	x_1	x_2	x_3	x_4	x_5	rhs	ratio test
iteration 2	z	1	2	-5	0	3	0	0	
basic variable	x_3	0	1	-1	1	1	0	0	$0^+/(-1) = 0^-$
	x_5	0	1	2	0	-1	1	4	$4/2 = 2 > 0$

Feasible solution: $(z, x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 0, 0, 4)$.
Optimality Test: No, since there is a negative coefficient for the z-equation.
Entering variable: x_2 for having the most negative z-equation coefficient.
Leaving variable: x_5 by the Ratio Test for being the only positive ratio.
Subsequent Row Operations: R2/2, 5*R2+R0, R2+R1.

		z	x_1	x_2	x_3	x_4	x_5	rhs	ratio test
iteration 3	z	1	9/2	0	0	1/2	5/2	10	
basic variable	x_3	0	3/2	0	1	1/2	1/2	2	
	x_2	0	1/2	1	0	-1/2	1/2	2	

Feasible solution: $(z, x_1, x_2, x_3, x_4, x_5) = (10, 0, 2, 2, 0, 2)$.
Optimality Test: Yes, since all z-equation's coefficients are non-negative. Optimal solution is obtained: $z^* = 10,\ (x_1^*, x_2^*, x_3^*) = (0, 2, 2)$.

(c) $E = \begin{bmatrix}1 & 1/2 & 5/2\\0 & 1/2 & 1/2\\0 & -1/2 & 1/2\end{bmatrix}$ because $ES_0 = \begin{bmatrix}1 & y^T\\0 & B\end{bmatrix}\begin{bmatrix}1 & -c^T & \mathbf{0} & b_0\\0 & A & I & b\end{bmatrix} = \begin{bmatrix}1 & -c^T + y^T A & y^T & b_0 + y^T b\\0 & BA & B & Bb\end{bmatrix} = S$

(d) Using the exact row operations, E, on the modified problem with the new right-hand side: $b = \begin{bmatrix}1\\4\end{bmatrix}$, the new feasible echelon form S has the same columns except for the last column $\begin{bmatrix}b_0 + y^T b\\Bb\end{bmatrix} = \begin{bmatrix}\text{0+(1/2)1+(5/2)4=10.5}\\\text{(1/2)1+(1/2)4=2.5}\\\text{(-1/2)1+(1/2)4=1.5}\end{bmatrix}$. Since the resulting right-hand is positive, the corresponding corner point solution is feasible. Because of the same optimality test holds, the new solution is the optimal solution for the modified problem. That is, the last feasible echelon form is:

		z	x_1	x_2	x_3	x_4	x_5	rhs	ratio test
	z	1	9/2	0	0	1/2	5/2	10.5	
basic variable	x_3	0	3/2	0	1	1/2	1/2	2.5	
	x_2	0	1/2	1	0	-1/2	1/2	1.5	

and the optimal solution is: $z^* = 10.5,\ (x_1^*, x_2^*, x_3^*) = (0, 1.5, 2.5)$.

3(35 pts) For a zero-sum game with the following payoff table

		Player 2		
Strategy		1	2	3
Player 1	1	3	2	-3
	2	-1	1	2

(a) Use the graphical method to find the optimal mixed strategies for Player 1 only.

(b) A fair game is one for which the game value $v = 0$. With everything else the same except for the pay-off for the {2,3}-play

		Player 2		
Strategy		1	2	3
Player 1	1	3	2	-3
	2	-1	1	a

use the graphical method to determine the parameter value a so that the game is a fair game.

(c) Use the graphical method to find the optimal mixed strategies for both players for the fair game (b).

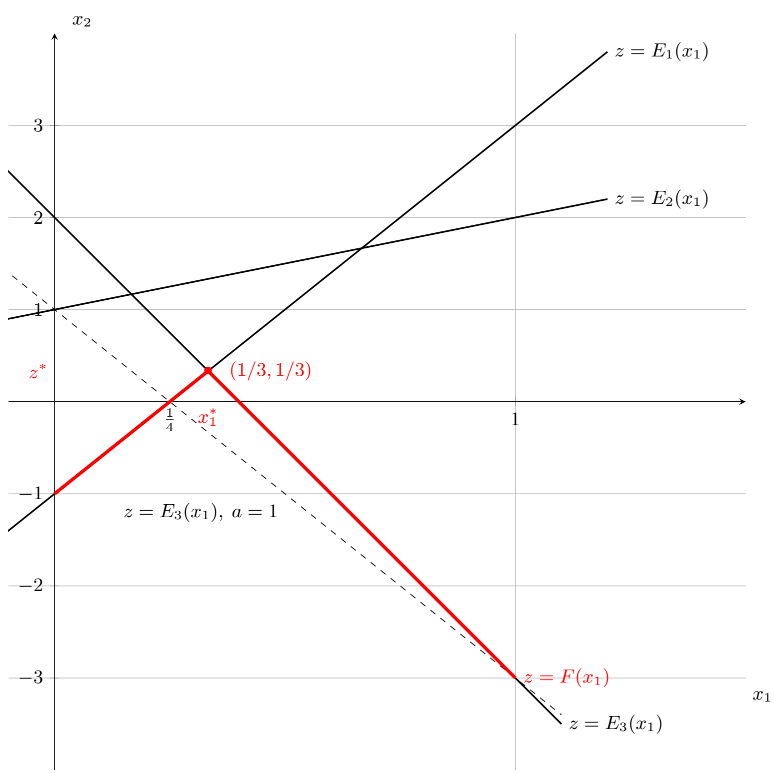
(d) Write down either the primal or the dual linear programming problem for the fair game, and use Excel/Solver to verify the solutions obtained above. Including screenshots for the input data sheet and the sensitivity report sheet from the Solver for supporting work.

Solution: (a) With $x_1 + x_2 = 1$ or $x_2 = 1 - x_1$, $0 \le x_1 \le 1$, we have

Y's pure strategies	**X's expected payoff**
$e_1 = (1,0,0)$:	$z = E_1(x_1) = 3x_1 - x_2 = 4x_1 - 1$
$e_2 = (0,1,0)$:	$z = E_2(x_1) = 2x_1 + x_1 = x_1 + 1$
$e_3 = (0,0,1)$:	$z = E_2(x_1) = -3x_1 + 2x_2 = -5x_1 + 2$

$$z = F(x_1) = \min_y E(x, y) = \min_y \{E_1(x_1)y_1 + E_2(x_1)y_2 + E_3(x_1)y_3\} = \min\{E_1(x_1), E_2(x_1), E_3(x_1)\}$$

The graphs of E_i and F are as follows:



From the graph, max $F(x_1)$ occurs at the intersection of $z = E_1(x_1)$ and $z = E_3(x_1)$. Solve the intersection point to obtain

$$(x_1^*, z^*) = \arg\max F(x_1) = (1/3, 1/3).$$

That, the optimal mixed strategies for X is $x^* = (1/3, 2/3)$ with optimal expected payoff $E^* = 1/3$.

(b) The parameter only affects E_3: $z = E_3(x_1) == -3x_1 + ax_2 = -3x_1 + a(1 - x_1)$, which goes through these points: $(0, a)$ and $(1, -3)$. For a fair game, it is from the graph that $z = E_1(x_1)$ must intersect the zero game value line $z = z^* = 0$ at $x_1^* = 1/4$, $x_2^* = 3/4$. As a result, $z = E_3(x_1)$ must goes through point $(x_1^*, z^*) = (1/4, 0)$ as well: $0 = E_3(1/4)$. Solving a from this equation to obtain:

$$0 = E_3(1/4) = -3/4 + a(3/4) \quad \rightarrow \quad a = 1.$$

(c) From (b) we already have $(x_1^*, x_2^*) = (1/4, 3/4)$. To solve for y, we know $y_2^* = 0$ because $z = E_2(x_1)$ lies strictly above the fair value point $(1/4, 0)$. At the optimal strategies, y^*, for Y, the expected payoff $z = E(x, y^*),\ a = 1$ must be the zero function

$$z = E(x, y^*) = E_1(x_1)y_1^* + E_3(x_1)y_3^* = (4x_1 - 1)y_1^* + (-4x_1 + 1)y_3^* \equiv E^* = 0.$$

Collect the like-terms above to have

$$-y_1^* + y_3^* + 4[y_1^* - y_3^*]x_1 \equiv 0.$$

The optimal solution y^* with $y_2^* = 0$ is the solution to the system of equations below:

$$\begin{aligned} -y_1^* + y_3^* &= 0 \\ y_1^* + y_3^* &= 1 \end{aligned}$$

Solve it to obtain $y^* = (1/2, 0, 1/2)$.

(d) The minimax problem is equivalent to the LP problem below

$$\begin{array}{ll} \text{Maximize} & z = x_3 - x_4 \\ \text{subject to} & 3x_1 - x_2 - x_3 + x_4 \geq 0 \\ & 2x_1 + x_2 - x_3 + x_4 \geq 0 \\ & -3x_1 + x_2 - x_3 + x_4 \geq 0 \\ & x_1 + x_2 = 1 \\ & x_i \geq 0,\ 1 \leq j \leq 4. \end{array}$$

The ExcelSolver shows the optimal solution is $(x_1, x_2) = (1/4, 3/4)$ for Player 1, $(y_1, y_2, y_3) = (1/2, 0, 1/2)$ for Player II from the Shadow Price, with the fair game value $v = x_3 - x_4 = 0 - 0 = 0$. The same as from (c).

x1	x2	x3	x4			
0	0	0	0			
0	0	1	-1	0		
3	-1	-1	1	0	>=	0
2	1	-1	1	0	>=	0
-3	1	-1	1	0	>=	0
1	1	0	0	0	"="	1

Microsoft Excel 16.0 Sensitivity Report
Worksheet: [worksheet.xlsx]WinDoor Example
Report Created: 6/21/2021 2:37:54 PM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B28	x1	0.25	0	0	4	4
C28	x2	0.75	0	0	4	4
D28	x3	0	0	1	0	1E+30
E28	x4	0	0	-1	0	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F32	<=	-1.11022E-16	-0.5	0	6	0
F33	<=	1.25	0	0	1.25	1E+30
F34	<=	1.11022E-16	-0.5	0	2	0
F35	<=	1	0	1	1E+30	1

Figure 1: #3 (d)