16.2-1 a) Since the probability of weather tomorrow is only dependent on today, Markovian property holds for evolution of the weather.

b) Let 0 = Rain, 1 = Clear,

Then $P = P^{(1)} = \begin{array}{c|cc} & 0 & 1 \\ 0 \\ 1 \end{array}\begin{bmatrix} .5 & .5 \\ .1 & .9 \end{bmatrix}$

16.2-2 a) Let 1 = increased today and yesterday
2 = increased today but decreased yesterday
3 = decreased today but increased yesterday
4 = decreased today and yesterday

Then $P = P^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \alpha_1 & 0 & 1-\alpha_1 & 0 \\ \alpha_2 & 0 & 1-\alpha_2 & 0 \\ 0 & \alpha_3 & 0 & 1-\alpha_3 \\ 0 & \alpha_4 & 0 & 1-\alpha_4 \end{bmatrix} \end{array}$

b) Because the state space is properly defined as (change today, change yesterday), and tomorrow's state will have its second component equal to today's first component.

16.3-1
a)
$$P^2 = \begin{bmatrix} 0.3 & 0.7 \\ 0.14 & 0.86 \end{bmatrix} \qquad P^5 = \begin{bmatrix} 0.175 & 0.825 \\ 0.165 & 0.835 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.167 & 0.833 \\ 0.167 & 0.833 \end{bmatrix} \qquad P^{20} = \begin{bmatrix} 0.167 & 0.833 \\ 0.167 & 0.833 \end{bmatrix}$$

(b) $P(\text{rain } n \text{ days from now} \mid \text{rain today}) = P_{11}^{n}$

If the probability it will rain today is 0.5,

$P(\text{rain } n \text{days from now}) = P_n = 0.5 \cdot P_{11}^{n}$

Thus. $P_2 = 0.15$, $P_5 = 0.0875$, $P_{10} = 0.0835$, $P_{20} = 0.0835$

(c) We find $\pi_1 = 0.167$
$\pi_2 = 0.833$

Obviously when n grows large, every row of $P^{(n)}$ approaches to the stationary probabilies.

16.3-3 a) Transition Matrix, $P = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{bmatrix}$

(b)

$$P^5 = \begin{bmatrix} 0.062 & 0.312 & 0.156 & 0.156 & 0.312 \\ 0.312 & 0.062 & 0.312 & 0.156 & 0.156 \\ 0.156 & 0.312 & 0.062 & 0.312 & 0.156 \\ 0.156 & 0.156 & 0.312 & 0.062 & 0.312 \\ 0.312 & 0.156 & 0.156 & 0.312 & 0.062 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.248 & 0.161 & 0.215 & 0.215 & 0.161 \\ 0.161 & 0.248 & 0.161 & 0.215 & 0.215 \\ 0.215 & 0.161 & 0.248 & 0.161 & 0.215 \\ 0.215 & 0.215 & 0.161 & 0.248 & 0.161 \\ 0.161 & 0.215 & 0.215 & 0.161 & 0.248 \end{bmatrix}$$

$$P^{20} = \begin{bmatrix} 0.206 & 0.195 & 0.202 & 0.202 & 0.195 \\ 0.195 & 0.206 & 0.195 & 0.202 & 0.202 \\ 0.202 & 0.195 & 0.206 & 0.195 & 0.202 \\ 0.202 & 0.202 & 0.195 & 0.206 & 0.195 \\ 0.195 & 0.202 & 0.202 & 0.195 & 0.206 \end{bmatrix}$$

$$P^{40} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

$$P^{80} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

(c) We can easily find out $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 0.2$

16.4-1

(a) P has one recurrent communicating class: $\{0, 1, 2, 3\}$

(b) P has three communicating classes:

$\{0\}$ is absorbing and, therefore, recurrent

$\{1, 2\}$ is recurrent

$\{3\}$ is transient

16.4-2

(a) P has one recurrent communicating class: $\{0, 1, 2, 3\}$

(b) P has one recurrent communicating class: $\{0, 1, 2\}$

16.5-1 $$P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$$

From $\pi P = P \Rightarrow$ $\alpha \pi_1 + (1-\beta)\pi_2 = \pi_1$

$\pi_1 + \pi_2 = 1$

$$\Rightarrow \pi = \left(\frac{1-\beta}{2-\alpha-\beta}, \frac{1-\alpha}{2-\alpha-\beta} \right)$$

16.5-4 Run the OR courseware and we can find

$\pi_1 = 0.346$

$\pi_2 = 0.385$

$\pi_3 = 0.269$

which means the steady-state market share for A and B are 0.346 and 0.385 respectively.

16.5-5

(a) Assume demand occurs after the delivery:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} .6 & .4 & 0 & 0 & 0 & 0 & 0 \\ .3 & .3 & .4 & 0 & 0 & 0 & 0 \\ .1 & .2 & .3 & .4 & 0 & 0 & 0 \\ 0 & .1 & .2 & .3 & .4 & 0 & 0 \\ 0 & 0 & .1 & .2 & .3 & .4 & 0 \\ 0 & 0 & 0 & .1 & .2 & .3 & .4 \\ 0 & 0 & 0 & 0 & .1 & .2 & .7 \end{bmatrix} \end{array}$$

16.5-5 (CONT'D)

(b) $\pi P = \pi \Rightarrow (.139, .139, .139, .138, .141, .130, .174)$

(C) The steady probability that a pint of blood is to be discarded

$$= P\{D=0\} \cdot P\{\text{state is in } 7\} = 0.4 \times 0.174 = 0.0696$$

(d) P{emergency deliveries}

$$= \sum_{i=1}^{2} P\{\text{state } i\} \cdot P\{D > i\}$$

$$= 0.139 \times (0.2 + 0.1) + 0.139 \times 0.1 = 0.0556$$

16.6-2 (a) Let 0 = operational, 1 = down, 2 = repaired

Then

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0 & 1 \\ 0.9 & 0.1 & 0 \end{bmatrix} \end{array}$$

(b) From sec 14.5, we have $\mu_{ij} = 1 + \sum_{k \neq j} p_{ik} \mu_{kj}$

So, $\begin{cases} \mu_{00} = 1 + 0.1\mu_{10} \\ \mu_{10} = 1 + \mu_{20} \\ \mu_{20} = 1 + 0.1\mu_{10} \end{cases}$ such that $\begin{cases} \mu_{00} = \frac{11}{9} \\ \mu_{10} = \frac{20}{9} \\ \mu_{20} = \frac{11}{9} \end{cases}$

$\begin{cases} \mu_{01} = 1 + 0.9\mu_{01} \\ \mu_{11} = 1 + \mu_{21} \\ \mu_{21} = 1 + 0.9\mu_{01} \end{cases}$ such that $\begin{cases} \mu_{01} = 10 \\ \mu_{11} = 11 \\ \mu_{21} = 10 \end{cases}$

$\begin{cases} \mu_{02} = 1 + 0.9\mu_{02} + 0.1\mu_{12} \\ \mu_{12} = 1 + 0 \\ \mu_{22} = 1 + 0.9\mu_{02} + 0.1\mu_{12} \end{cases}$ such that $\begin{cases} \mu_{02} = 11 \\ \mu_{12} = 1 \\ \mu_{22} = 2.09 \end{cases}$

The expected number of days that machine will remain operational is 10.

(C) It would be the same due to the Markovian Property.

16.3-2 (a) Let states 0 and 1 denote that a 0 and a 1 have been recorded, respectively.

$$P = \begin{bmatrix} 1-q & q \\ q & 1-q \end{bmatrix} \quad \text{where } q = 0.01$$

(b) $$P^{(10)} = \begin{bmatrix} 0.909 & 0.091 \\ 0.091 & 0.909 \end{bmatrix}$$

The probability that a digit would recorded accurately is 0.909

(c) $$P^{(10)} = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{bmatrix}$$

The probability is 0.99.

16.5-8

$$x_{t+1} = \begin{cases} \max\{x_t+2-D_{t+1}, 0\} & \text{for } x_t \le 1 \\ \max\{x_t - D_{t+1}, 0\} & \text{for } x_t \ge 2 \end{cases}$$

$$\text{So } P = \begin{bmatrix} .264 & .368 & .368 & 0 \\ .080 & .184 & .368 & .368 \\ .264 & .368 & .368 & 0 \\ .080 & .184 & .368 & .368 \end{bmatrix}$$

Solving the steady state equations gives $\pi_0 = .182$, $\pi_1 = .285$, $\pi_2 = .368$, $\pi_3 = .165$

b) $\lim_{n\to\infty} E\left(\frac{1}{n}\sum_{t=1}^{n} C(X_t)\right) = 0\pi_0 + 2\pi_1 + 8\pi_2 + 18\pi_3 = 6.48$

16.6-1

(a) The process satisfies the four points listed in Sec. 15.3, and is thus Markovian.

Let 0 = Working

1 = Failed

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & .9 & .1 \\ 1 & .65 & .35 \end{array}$$

b) $\mu_{00} = 1 + .1\mu_{10} \to \mu_{00} = 1.15$

$\mu_{01} = 1 + .9\mu_{01} \to \mu_{01} = 10$

$\mu_{10} = 1 + .35\mu_{10} \to \mu_{10} = 1.54$

$\mu_{11} = 1 + .65\mu_{01} \to \mu_{11} = 7.5$

16.6-5

(a) Transition Matrix, $P = \begin{bmatrix} 0 & 0.875 & 0.062 & 0.062 \\ 0 & 0.75 & 0.125 & 0.125 \\ 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Solution:

$\pi_1 = 0.154$
$\pi_2 = 0.538$
$\pi_3 = 0.154$
$\pi_4 = 0.154$

(b) $\pi \cdot C = 1(0.538) + 3(0.154) + 6(0.154) = \$1923.08$

(c)
$$\begin{cases} \mu_{00} = 1 + 0.875\mu_{10} + 0.0625\mu_{20} + 0.0625\mu_{30} \\ \mu_{10} = 1 + 0.75\mu_{10} + 0.125\mu_{20} + 0.125\mu_{30} \\ \mu_{20} = 1 + 0.5\mu_{20} + 0.5\mu_{30} \\ \mu_{30} = 0 + 1 \end{cases}$$

$\Rightarrow \mu_{00} = 6.5$, which is expected recurren time for state 0.