Prototypical Problem

max: $z = c_1x_1 + \cdots + c_nx_n$

sub. to:
$$(*)\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \le b_1 \\ \quad\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \le b_m \\ x_i \ge 0,\ 1 \le i \le n \end{cases}$$

Equivalent Problem

max: $z = c_1x_1 + \cdots + c_nx_n$

sub. to.
$$(*)\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\ \quad\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + \cdots x_{n+m} = b_m \\ x_i \ge 0,\ x_{n+j} \ge 0,\ 1 \le i \le n,\ 1 \le j \le m. \end{cases}$$

Properties:

- Constraint (*) defines an n-d simplex in (n+m)-d space in $(x_1, \cdots, x_{n+m})$
- Max. of z occurs at vertex, a corner point of the simplex.
- The simplex is bounded by (n+m-1)-d hyperplanes, and each hyperplane is defined by the equation $x_i = 0$ for some $i = 1, 2, \cdots, n, n+1, \cdots, n+m$.
- Each vertex or corner point of the n-d simplex (feasible region) is the intersection of n hyperplanes, i.e., $x_{i_1} = x_{i_2} = \cdots = x_{i_n} = 0$ for $1 \le i_k \le n+m$. Its complete characterization, $(x_1, x_2, \cdots, x_n)$, requires finding the remaining values, for x_j with $j \ne i_k$, from the system of the m constraint equations (*)
- The method of Gausian elimination is used to solve the m constraint equations (*) for x_j, $j \ne i_k$, with $x_{i_1} = x_{i_2} = \cdots = x_{i_n} = 0$.
- The simplex method is an iterative method ~~for~~ by which at each iteration it is to determine which corner point is to determine, or which n variables should be assigned to zero (or the status of nonbasic variables) and then solve for the remaining m variables from the m many constraint equations.

Simplex Method (algebraic Form and Tabular Form) ②

Standard (or canonical) Form.

max: $z - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\ \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + \cdots + x_{n+m} = b_m \end{cases}$$

Initial Corner Point Feasible: $x_1 = x_2 = \cdots = x_n = 0$, $x_{n+1} = b_1, \cdots, x_{n+m} = b_m$

i.e. $(0, \cdots, 0, b_1, \cdots, b_m)$ w/ z-value : $z = 0$

Use Gausian Elimination to derive the kth canonical form.

max. $z + c_{1,k}x_1 + c_{2,k}x_2 + \cdots + c_{n+m,k}x_{n+m} = b_{0,k}$

$$a_{11,k}x_1 + a_{12,k}x_2 + \cdots + a_{1m,k}x_{n+m} = b_{1,k}$$
$$\vdots$$
$$a_{m1,k}x_1 + a_{m2,k}x_2 + \cdots + a_{mn,k}x_{n+m} = b_{m,k}$$

w/ n hyperplanes (or nonbasic variables): $x_{i_1} = x_{i_2} = \cdots = x_{i_n} = 0$

The form is canonical if (~~through~~ by Gausian Edimiation) the following conditions are satisfied:

(1) All the coefficients of the basic variables in ~~the z-equ~~ the z-equation are zero: $c_{j,k} = 0$ for $j \neq i_\ell$, $1 \le \ell \le n$.

(2) For each basic variable x_j, $1 \le j \le m$, $j \neq i_\ell$, $1 \le \ell \le n$, there is one and only one constraint equation, j_r, $1 \le r \le m$, so that all the coefficients of the basic variables are zero except for the coefficient of x_j which is 1, i.e. $a_{j_r j,k} = 1$, and $a_{j_r p,k} = 0$, $p \neq j$, $p \neq i_\ell$, $1 \le \ell \le n$.

(3) ~~As a result, the~~ All right hand values are ~~possible~~ non-negative, $b_{i,k} \ge 0$, $0 \le i \le m$.

(4) As a result, the corner point feasible: $x_{i_1} = x_{i_2} = \cdots = x_{i_n} = 0$, $x_j = b_{j_r,k} \ge 0$, $j \neq i_\ell$, with z-value: $z = b_{0,k}$.

Criteria to stop the iterative process of simplex method.

- At the kth iterative canonical form

$$z + c_{1,k}x_1 + c_{2,k}x_2 + \cdots + c_{n+m,k}x_{n+m} = b_{0,k}$$

with nonbasic variables $x_{i_1} = x_{i_2} = \cdots = x_{i_n} = 0$, and basic variables $x_j > 0$ ~~with $j \neq i_\ell$, $1 \le \ell \le n$,~~ for which $c_{j,k} = 0$ with $j \neq i_\ell$, $1 \le \ell \le n$, if all the coefficients, $c_{i_\ell,k}$, for the nonbasic variables are nonnegative, then the kth z-value, $b_{0,k}$, is the maximum because it cannot be improved further by increasing any x_{i_ℓ} from 0. So in this case, optimal solution is found, and stop the method.

- If there is ~~at least one coefficient~~ a negative coefficient, say $c_{i_\ell,k} < 0$, for at least one nonbasic variable x_{i_ℓ}, the simplex method need continue for one more step.

- Here is the algorithm to determine the new set of nonbasic variables (i.e. new set of n hyperplanes for the new corner point which is adjacent to the old one).

- Let i_ℓ be the index so that the coefficient $c_{i_\ell,k}$ is the most negative so that the new (i.e. (k+1)st) set of nonbasic variables must NOT include x_{i_ℓ}.

- To determine the new set of nonbasic variables, we must find a variable from the old (i.e. kth) basic variables x_j, $j \neq i_0$, $1 \le 0 \le n$, to take the place of x_{i_ℓ}. This x_j is determined by the Ratio Test: Recall from the kth canonical form that the x_j, $j \neq i_0$, is solved from the j_rth equation of the constraints. The the new nonbasic variable is this x_j so that $b_{j_r,k}/a_{j_r,k}$ is the smallest for $1 \le r \le m$. The new basic variable x_{i_ℓ} now corresponds to the j_rth equation in the (k+1)st iteration of the Gausian elimination.