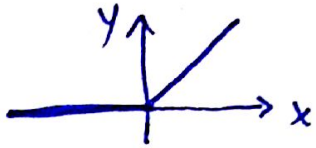
Review: Queueing Theory.

Key parameters & variables.

- n — Queue or ~~sto~~ system state, # of cust. in the system.
- P_n — probability dist.
- L — mean length of the system
- L_q — mean length of the queue.
- s — # of servers.
- $\bar{\lambda}$ — mean arrival rate
- $\bar{\mu}$ — mean service rate of one server
  ($s\bar{\mu}$ — mean service rate of the system.)
- W — mean time in the systems
- W_q — mean time in the queue.
- T — random variable of interarrival or service completion time.
- $Z(t)$ — random variable of arrivals or service completion in $[0, t]$.

- Key relationships.



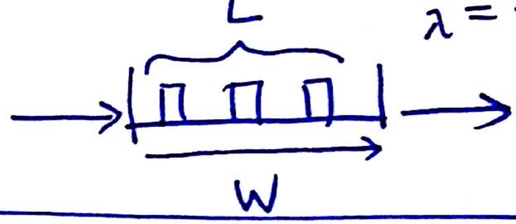
$$L = E(n) = \sum_{n=0}^{\infty} n P_n$$

$$L_q = E([n-s]_+) = \sum_{n=s}^{\infty} (n-s) P_n = \sum_{n=0}^{\infty} [n-s]_+ P_n , \quad [x]_+ = \begin{cases} x, & \text{if } x>0 \\ 0, & \text{if } x \le 0 \end{cases}$$

$$\rho = \frac{\bar{\lambda}}{s\bar{\mu}}$$ — utilization parameter.

$$\bar{\lambda} = \frac{L}{W}$$

$$W = \frac{L}{\bar{\lambda}}, \quad W_q = \frac{L_q}{\bar{\lambda}}, \quad \bar{\mu} = \frac{1}{W - W_q}$$



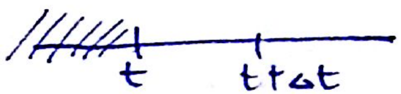
- Poisson v.s. Exponential Distribution.

- $P\{X(t) = n\} = \dfrac{(\alpha t)^n e^{-\alpha t}}{n!}, \quad n = 0, 1, 2, \dots$

$$\bar{\lambda} = \frac{E(X(t))}{t}$$

- $E(X(t)) = \sum_{n=0}^{\infty} n P\{X(t) = n\} = \alpha t$ $\qquad \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = e^x$

- $\boxed{P\{T > t\} = P\{X(t) = 0\} = e^{-\alpha t}}$



w/ pdf:

$$f_T(t) = \lim_{\Delta t \to 0} \frac{P\{t < T < t + \Delta t\}}{\Delta t} = \lim_{\Delta t \to 0} \frac{P(T>t) - P(T > t+\Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{e^{-\alpha t} - e^{-\alpha(t+\Delta t)}}{\Delta t} = +\alpha e^{-\alpha t}, \quad t \ge 0.$$

- Aggregation property: $T_1, \cdots, T_k$ are exp. dist. w/ parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$. and independent. Then. $T = \min_{1 \le i \le k} \{T_i\}$,

$$P\{T > t\} = e^{-(\alpha_1 + \cdots + \alpha_k)t}.$$

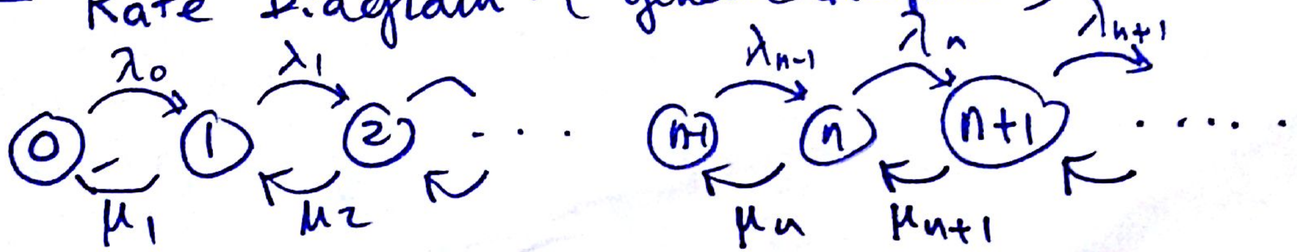
- $E(T) = \frac{1}{\alpha}$
- Lack of memory.
- Same for Poisson distribution.

- Birth-Death Process.
  - Expt. dirt. for both input & output ~~prod~~ processes.
  - Indep. input & output process.
  - Rate Balance Equation. for steady state process

$$\boxed{\text{Input rate to state } n = \text{output rate from state } n}$$

based on. $|E_n(t) - L_n(t)| \le C$

- Applications to specific situations.
  — Rate Diagram. (generic diagram)



- Dist. P_n :

$$C_0 = 1, \quad C_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_1 \boxed{\lambda_0}}{\mu_n \mu_{n-1}\cdots\mu_2 \boxed{\mu_1}},$$

$$P_0 = \frac{1}{\sum_{n=0}^{\infty} C_n} = \frac{1}{C_0 + C_1 + \cdots + C_n + \cdots}$$ ~~pro~~ provided existing.

$$P_n = C_n P_0 = \frac{C_n}{C_0 + C_1 + \cdots + C_n + \cdots}$$

- M/M/1 model. $\lambda_n = \lambda, \quad \mu_n = \mu.$

$$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}, \quad |x| < 1$$

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2}$$

$$\Rightarrow L = \frac{\rho}{1-\rho}, \quad \rho = \frac{\lambda}{\mu}, \quad \text{etc.}$$

Priority Queue.

- W_{1-k}
- W_{2-k}
- $\vdots$

- M/G/1. G — general dist. for service completion w/ $E(T) = \mu$, $var(T) = \sigma^2$. $\Rightarrow \ldots$

- Erlang Dist. for service completion : $T = T_1 + T_2 + \cdots + T_k$ w/ exp. dist. for $T_1, T_2, \ldots, T_k$ w/ $\mu_i = \mu$. $\Rightarrow P(T > t)$ is Erlang Dist