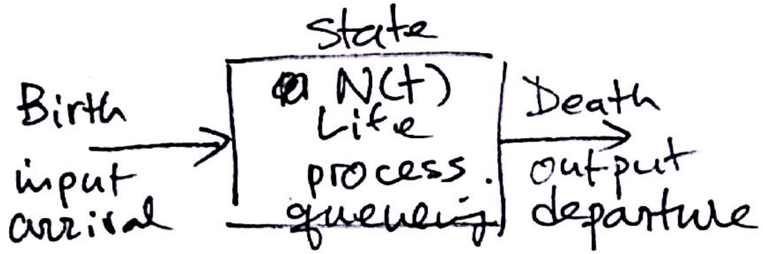
§17.5. Birth-&-Death Process.

(1)

Consider a general b-d process.



t — time

state $= \{0, 1, 2, 3, \dots\}$

$N(t) = n$

Assume the following:

1. Given the state $N(t) = n$, the birth time interval is exponentially distributed with rate λ_n

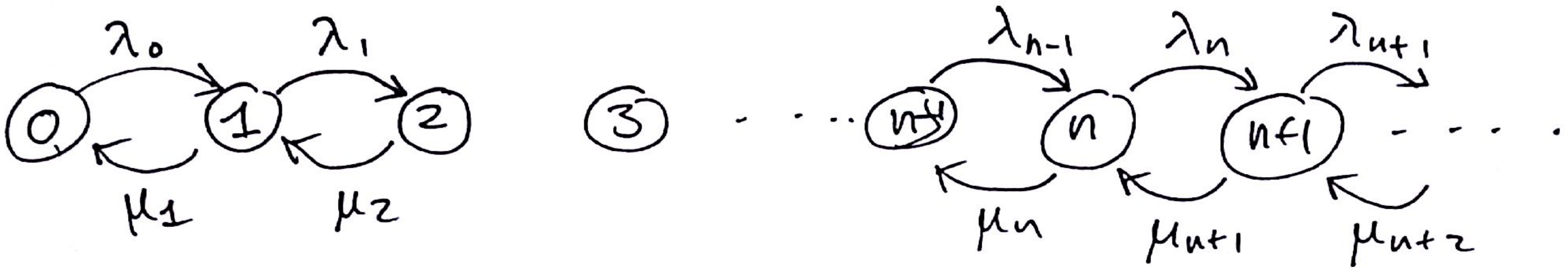
   $\left(f_T(t) = \begin{cases} \lambda_n e^{-\lambda_n t}, & t > 0 \\ 0 & t \le 0 \end{cases}, \quad E(T) = \frac{1}{\lambda_n}, \quad E(N(t)) = \lambda_n t \right)$

2. Given the state $N(t) = n$, the death time interval is exponentially distributed with rate μ_n

3. With short enough sampling time interval (Δt) (or observation time interval), the transition of state $N(t) = n$ to the next state $N(t + \Delta t) = n+1$ or $n-1$.

4. All birth & death events are independent.

Rate Diagram.



Consider steady-state process.

Def: $E_n(t)$ = # of times the process enters state n ~~to~~ between time 0 and t

$L_n(t)$ = # of times the process leaves state n between 0 and t.

Need to know the average rate

$$\frac{E_n(t)}{t}, \quad \lim_{t\to\infty} \frac{E_n(t)}{t}, \quad \frac{L_n(t)}{t}, \quad \lim_{t\to\infty} \frac{L_n(t)}{t}.$$

Because every entering event into state n must be paired with a leaving event before another entering. We have

$$|E_n(t) - L_n(t)| \leq 1$$

$$\Rightarrow \left| \frac{E_n(t)}{t} - \frac{L_n(t)}{t} \right| \leq \frac{1}{t}$$

$$\Rightarrow \left| \lim_{t\to\infty} \frac{E_n(t)}{t} - \lim_{t\to\infty} \frac{L_n(t)}{t} \right| \leq 0.$$

$$\Rightarrow \bar{\lambda}_n = \lim_{t\to\infty} \frac{E_n(t)}{t} = \lim_{t\to\infty} \frac{L_n(t)}{t}. = \bar{\mu}_n$$

$\boxed{\text{Mean birth-rate} = \text{mean death-rate.}}$ ~~for all state,~~ ∴ for steady state process.

Let P_n be the probability ~~dist~~ distribution for the the state $= n$.

$$\bar{\lambda}_0 = \sum_{n=0}^{\infty} [\text{rate from state } n \text{ to state } 0] P_n$$

$$= \mu_1 P_1 \quad \text{by the rate diagram since } \text{rate}_{n\to 0} = 0, \text{ except } n=1$$

$$\bar{\mu}_1 = \left(\sum_{n=0}^{\infty} [\text{~~rate~~ death rate from 1 to state } n\right) P_0 .$$