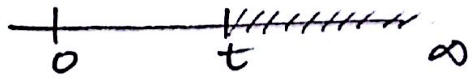
By calculus,

- $P(T \leq t) = \int_0^t f_T(s)\,ds = \int_0^t \alpha e^{-\alpha s}\,ds = 1 - e^{-\alpha t}$

$P(T < +\infty) = \lim_{t \to \infty} P(T \leq t) = \lim_{t \to \infty} (1 - e^{-\alpha t}) = 1.$

$P(T > t) = P(T < \infty) - P(t \leq t)$

$= 1 - (1 - e^{\alpha t}) = e^{-\alpha t}$



- $\bar{T} = E(T) = \int_0^\infty t \underbrace{f_T(t)\,dt} = \int_0^\infty \alpha t e^{-\alpha t}\,dt = \frac{1}{\alpha}$

$\Leftrightarrow \alpha = \frac{1}{\bar{T}} = \bar{\lambda}$ — mean arrival rate (#cust./time)

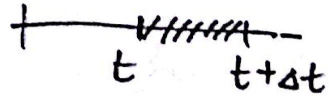
$var(T) = \int_0^\infty (t - \frac{1}{\alpha})^2 \underbrace{f_T(t)\,dt} = \frac{1}{\alpha^2}$

- <u>Properties</u>:
  - $P(0 \leq T \leq \Delta t) \geq P(t \leq T \leq t + \Delta t)$ since

$P(0 \leq T \leq \Delta t) = 1 - e^{-\alpha(\Delta t)}$

$P(t \leq T \leq t + \Delta t) = P(T \leq t + \Delta t) - P(T \leq t)$

$= 1 - e^{-\alpha(t + \Delta t)} - (1 - e^{-\alpha t})$



$$= e^{-\alpha t} - e^{-\alpha t - \alpha \Delta t}$$

$$= e^{-\alpha t}\underbrace{(1 - e^{-\alpha \Delta t})} = e^{-\alpha t} P(T \leq \Delta t)$$

$$\leq P(T \leq \Delta t).$$

- Lack of memory:

$$P(T > t + \Delta t \mid T > \Delta t)$$

$$= \frac{P(T \geq t + \Delta t,\ T > \Delta t)}{P(T > \Delta t)} = \frac{P(T \geq t + \Delta t)}{P(T > \Delta t)} = \frac{e^{-\alpha(t+\Delta t)}}{e^{-\alpha \Delta t}}$$

$$= e^{-\alpha t} = P(T > t)$$

- $$P(T < t + \Delta t \mid T > t) = P(t < T < t + \Delta t) = P(T < \Delta t).$$

$$= 1 - e^{-\alpha \Delta t} \doteq 1 - (1 - \alpha \Delta t) = \alpha \Delta t.$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \cdots$$

- The minimum of a finite independent distribution of exponential random variable is also exponentially distributed.

$T_1, T_2, \cdots, T_n$ — random variables, exponentially distributed w/ parameters $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Let $T = \min\{T_1, T_2, T_3, \cdots, T_n\}$ — random variable.

Then

$$P(T > t) = P(T_1, T_2, \cdots, T_n > t)$$
$$= P(T_1 > t)\, P(T_2 > t) \cdots P(T_n > t)$$
$$= e^{-\alpha_1 t} \cdot e^{-\alpha_2 t} \cdots e^{-\alpha_n t}$$
$$= e^{-(\alpha_1 + \alpha_2 + \cdots + \alpha_n) t} := e^{-\alpha t}$$

$\Rightarrow$ T is of exp. dist. w/ parameter $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Property: T — exp. dist. w/ parameter α.

Let $\delta(t)$ be the random variable for the number of event (arrival, or completion, etc) by time t.

Then

$$P\{\delta(t) = n\} = \frac{(\alpha t)^n e^{-\alpha t}}{n!}, \quad n = 0, 1, 2, \cdots.$$

(The Poisson distribution).

Special case: $P\{\delta(t) = 0\} = P(T > t) = e^{-\alpha t}$

Ex: A 4-server queueing system has a controlled arrival process that provides customers in time to keep the servers continuously busy. Service times have an exponential distribution of a mean service time 0.3 hr. All servers begin service at time $t=0$. The first completion occurs at $t_1 = 0.5$ hr. Determine the expected time after $t_1 = 0.5$ hr untill the next service is completed from the other servers

T_i — service completion time for server i, $i = 1, 2, 3, 4$.

$E(T_i) = \bar{T}_i = 0.3\,hr = \frac{1}{\alpha_i}$, $\Rightarrow \alpha_i = \frac{1}{0.3}\,/hr = \frac{10}{3}\,/hr$.

$h = 0.5\,hr = 30\,min$. by server 1,

$T = \min\{T_2, T_3, T_4\}$

Because of lack of memory for exponential distribution,

$E(T) = \bar{T} = \frac{1}{\alpha}$, with $\alpha = \alpha_2 + \alpha_3 + \alpha_4$

i.e. the aggregation over T_2, T_3, T_4.

$$\alpha = \alpha_2 + \alpha_3 + \alpha_4 = \frac{10}{3} + \frac{10}{3} + \frac{10}{3} = 10\,/hr.$$

$$E(T) = \frac{1}{\alpha} = \frac{1}{10} = 0.1\,hr = 6\,min.$$

$P(T > t+h \mid T > h) = e^{-\alpha t}$ indep. of h to the conditional prob. dist. which is shown exponential.