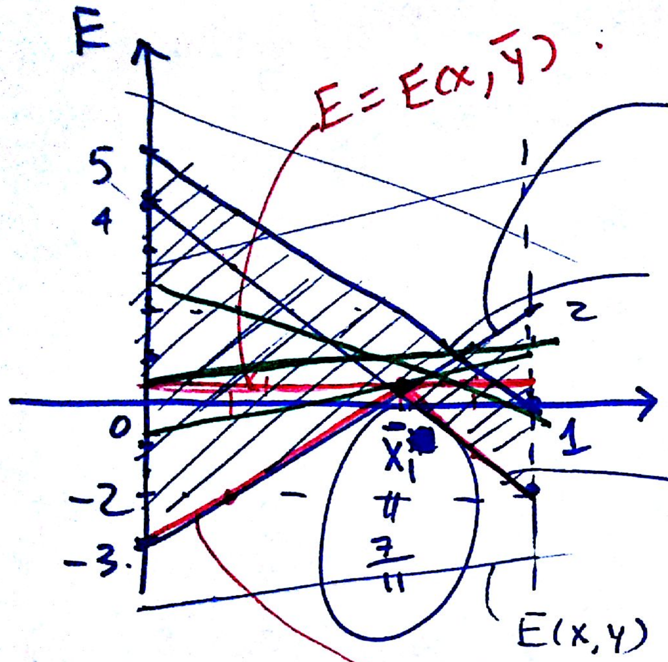


$E = E_{x_3}(x_1)$, $\quad y = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$= -3+5x_1$

$E = E_{x_1}(x_1) = 5(1-x_1)$, $\quad y = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$E = E(x,y) = \sum y_i\, E_{x_i}(x_1)$ — a function of $x_1$ to each fixed $y$. It is a line in $x_1$

$E = E_{x_2}(x_1) = 4-6x_1$, $\quad y = \begin{bmatrix}0\\1\\0\end{bmatrix}$

Sketch $F(x) = \min\limits_{y} E(x,y) = \min\limits_{1\le j\le 3}\{E_{x_j}(x_1)\}$.

$E = F(x_1) = \min\limits_{y} E(x,y)$.

$$E(\bar{x},\bar{y}) = \max_{x}\left(\min_{y} E(x,y)\right)$$

$$= \max_{0\le x_1\le 1}\left(F(x_1)\right) = F(x_1^*) \overset{a}{=} F(\bar{x}_1).$$

$\bar{x}_1$ is the intersection of $\begin{cases} E = E_{x_3}(x_1) = -3+5x_1 \\ E = E_{x_2}(x_1) = 4-6x_1 \end{cases}$.

$\Rightarrow -3+5x_1 = 4-6x_1 \Rightarrow \bar{x}_1 = \frac{7}{11}$. $\Rightarrow \bar{x} = \left(\frac{7}{11}, \frac{4}{11}\right)$

of which $E(\bar{x},\bar{y}) = -3+5\left(\frac{7}{11}\right) = \frac{2}{11}$

Next to find the optimal solution $\bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3)$, at which

$$E(x, \bar{y}) = \bar{y}_1 E_{x_1}(x_1) + \bar{y}_2 E_{x_2}(x_1) + \bar{y}_2 E_{x_3}(x_1), \text{ a function of } x_1$$

$$= (5\bar{y}_1 + 4\bar{y}_2 - 3\bar{y}_3) + (-5\bar{y}_1 - 6\bar{y}_2 + 5\bar{y}_3)x_1$$

Is a line, & has an interior maximum ~~at~~ $\bar{x}_1$, at which

$$E(\bar{x}, \bar{y}) = \max_x E(x, \bar{y})$$

$\Rightarrow$. $E(x, \bar{y})$ must be a horizontal line

$$E(x, \bar{y}) \equiv E(\bar{x}, \bar{y}) = \frac{2}{11}.$$

Hence:
$$\begin{cases} 5\bar{y}_1 + 4\bar{y}_2 - 3\bar{y}_3 = \frac{2}{11} & (1) \\ -5\bar{y}_1 - 6\bar{y}_2 + 5\bar{y}_3 = 0 & (2) \\ \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1 & (3). \end{cases}$$

Solve it: $\bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, \frac{5}{11}, \frac{6}{11})$. $\square$.

**Thm**: If ~~Ga~~, $\bar{x}$ is a solution to LP: $\min. z = v$

$$\text{sub. to } x^T A \le v[1, 1, \dots, 1], \quad x_i \ge 0, \ \Sigma x_i = 1$$

and $\bar{y}$ is a solution to the dual LP: $\max.\ w = u$

$$Ay \ge u\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}, \quad y_i \ge 0, \ \Sigma y_i = 1$$

(for which both have the same optimal value $\bar{v} = \bar{u}$),

then $(\bar{x}, \bar{y})$ must be an optimal soln. to the game problem with $A$ being the payoff matrix for player $\underline{X}$.

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- By simplex method, to find $(\bar{x}, \bar{y})$ for the game problem,

solve: $\max: w = u$, sub. $Ay \ge u\vec{1}$, $y_i \ge 0, \ \Sigma y_i = 1$. to get $\bar{y}$, and $\bar{x}$ is the

<u>shadow price</u> of this LP, (Lagrange multiplier), $(\bar{x}, \bar{u})$ )

- $u$ - may not be positive.

- let $\bar{u} = \min_{i,j} \{a_{ij}\}$. let $c = \min\{0, \bar{u}\} = \begin{cases} 0, & \text{if } \bar{u} \geq 0 \\ \bar{u}, & \text{if } \bar{u} < 0 \end{cases}$

Then consider the game problem w/ $A - c$ payoff matrix, so that $\min\{a_{ij} - c\} \geq 0$. $\Rightarrow$ game value $u$, or $v$ $\geq 0$.

- **Alt**: let $u = u_1 - u_2$, w/ $u_1 \geq 0$, $u_2 \geq 0$.

max: $w = u_1 - u_2$

sub. $Ay \geq u_1 \vec{1} - u_2 \vec{1}$

$y_i \geq 0, \sum y_i = 1, \quad u_1, u_2 \geq 0$.

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Ex: Find the optimal strategy for the following game by simplex method.

| | | P2 | |
|---|---|---|---|
| | | $y_1$ 1 | $y_2$ 2 |
| $P_1$ | $x_1$ 1 | -2 | 2 |
| | $x_2$ 2 | 4 | -3 |

For player Y.

Soln: To make sure its game value is positive, shift the payoff by +3 and consider the equivalent problem:

| | $y_1$ 1 | $y_2$ 2 |
|---|---|---|
| $x_1$ 1 | 1 | 5 |
| $x_2$ 2 | 7 | 0 |

The corresponding linear programming problem is

max: $z = x_3$

sub. to: $1x_1 + 7x_2 - x_3 \geq 0$

$5x_1 \qquad - x_3 \geq 0$

$x_i \geq 0, \; x_1 + x_2 = 1$

The equivalent dual problem is

min: $z = y_3$ or max $w = -y_3$

sub. to $1y_1 + 5y_2 - y_3 \leq 0$

$7y_1 \qquad - y_3 \leq 0$

$y_i \geq 0, \; y_1 + y_2 = 1$

Now solve the dual problem by the tabular form w/ big-M:

| basic var | w | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | RHS | Ratio Test |
|---|---|---|---|---|---|---|---|---|---|
| w | 1 | -M | -M | 1 | 0 | 0 | 0 | -M | Continue |
| $y_4$ | 0 | 1 | 5 | -1 | 1 | 0 | 0 | 0 | 0 |
| $y_5$ | 0 | 7 | 0 | -1 | 0 | 1 | 0 | 0 | 0 |
| $y_6$ | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| w | 1 | 0 | 4M | -M+1 | M | 0 | 0 | -M | Continue |
| $y_1$ | 0 | 1 | 5 | -1 | 1 | 0 | 0 | 0 | -0 |
| $y_5$ | 0 | 0 | -35 | 6 | -7 | 1 | 0 | 0 | 0 |
| $y_6$ | 0 | 0 | -4 | 1 | -1 | 0 | 1 | 1 | 1 |
| w | 1 | 0 | $-\frac{11}{6}M+\frac{35}{6}$ | 0 | $-\frac{1}{6}M+\frac{7}{6}$ | $\frac{1}{6}M+\frac{1}{6}$ | 0 | -M | Continue |
| $y_1$ | 0 | 1 | $-\frac{5}{6}$ | 0 | $-\frac{1}{6}$ | $\frac{1}{6}$ | 0 | 0 | -0 |
| $y_3$ | 0 | 0 | $-\frac{35}{6}$ | 1 | $-7/6$ | $1/6$ | 0 | 0 | -0 |
| $y_6$ | 0 | 0 | $\frac{11}{6}$ | 0 | $\frac{1}{6}$ | $-1/6$ | 1 | 1 | $\frac{6}{11}$ |
| w | 1 | 0 | 0 | 0 | $\frac{7}{11}$ | $\frac{4}{11}$ | $M-\frac{35}{11}$ | $-\frac{35}{11}$ | Stop |
| $y_1$ | 0 | 1 | 0 | 0 | $-\frac{1}{11}$ | $\frac{1}{11}$ | $\frac{5}{11}$ | $\frac{5}{11}$ | |
| $y_3$ | 0 | 0 | 0 | 1 | [illegible] | $-\frac{4}{11}$ | $\frac{35}{11}$ | $\frac{35}{11}$ | |
| $y_2$ | 0 | 0 | 1 | 0 | $\frac{1}{11}$ | $-\frac{1}{11}$ | $\frac{6}{11}$ | $\frac{6}{11}$ | |

→ Optimal solutions

$y_1^* = \frac{5}{11}$, $y_2^* = \frac{6}{11}$,

$x_1^* = \frac{7}{11}$, $x_2^* = \frac{4}{11}$

(shadow prices of dual problem)

w/ $w = -\frac{35}{11}$

or $z = -w = \frac{35}{11} (= y_3)$

The game value for the original problem is

$v = z - 3 = \frac{2}{11}$

by shifting 3 unit down.

Notice that the solution is the same as obtained by the graphical method