The goal of the game is ~~to~~ for player 1 to maximize $E(x,y)$ and for player 2 to maximize its return $-E(x,y)$ or is to minimize $E(x,y)$.

Goal is to find mixed strategy probability $(\bar{x}, \bar{y})$, with $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$,

$x_i \geq 0$, $\sum x_i = 1$, $y_j \geq 0$, $\sum y_j = 1$ so that

$$E(\bar{x}, \bar{y}) = \max_x E(x, \bar{y}) \text{ and}$$

$$E(\bar{x}, \bar{y}) = \min_y E(\bar{x}, y)$$

Such a point $(\bar{x}, \bar{y})$ is called the Nash equilibrium, or optimal game solution, $\bar{v} = E(\bar{x}, \bar{y})$ is called the game value.

Thm: If $(\bar{x}, \bar{y})$ is an NE, then

$$E(\bar{x}, \bar{y}) = \max_x \left(\min_y E(x,y) \right)$$

$$= \min_y \left(\max_x E(x,y) \right)$$

Graphical method: For games for which one player has only two strategies

Ex: Consider the game

		P2		
		y_1	y_2	y_3
P_1	x_1	0	-2	2
	x_2	5	4	-3

$\boxed{x_2 = 1 - x_1 \quad x_1 \geq 0}$

sum over columns or mixed strategy

Soln: $E(x, y) = \sum_{i,j} a_{ij} x_i y_j = \sum_i x_i \left(\sum_j a_{ij} y_j\right) = \sum_j y_j \left(\sum_i a_{ij} x_i\right)$

$= y_1(0x_1 + 5x_2) + y_2(-2x_1 + 4x_2) + y_3(2x_1 + (-3)x_2)$

$= y_1(0x_1 + 5(1-x_1)) + y_2(-2x_1 + 4(1-x_1)) + y_3(2x_1 + (-3)(1-x_1))$.

$= y_1[5(1-x_1)] + y_2[4 - 6x_1] + y_3[-3 + 5x_1]$.

$= y_1 E(x, \begin{bmatrix}1\\0\\0\end{bmatrix}) + y_2 E(x, \begin{bmatrix}0\\1\\0\end{bmatrix}) + y_3 E(x, \begin{bmatrix}0\\0\\1\end{bmatrix})$.

$= y_1 E_{x1}(x_1) + y_2 E_{x2}(x_1) + y_3 E_{x3}(x_1)$

Lemma: Consider $z = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k$, $w_i \geq 0$, $\sum w_i = 1$

Then

$$\max_{\substack{w\geq 0 \\ \sum w_i = 1}} \sum c_i w_i = \max_i \{c_i\}$$

$$\min_{\substack{w\geq 0 \\ \sum w_i = 1}} \sum c_i w_i = \min_j \{c_j\}.$$

Pf: Read

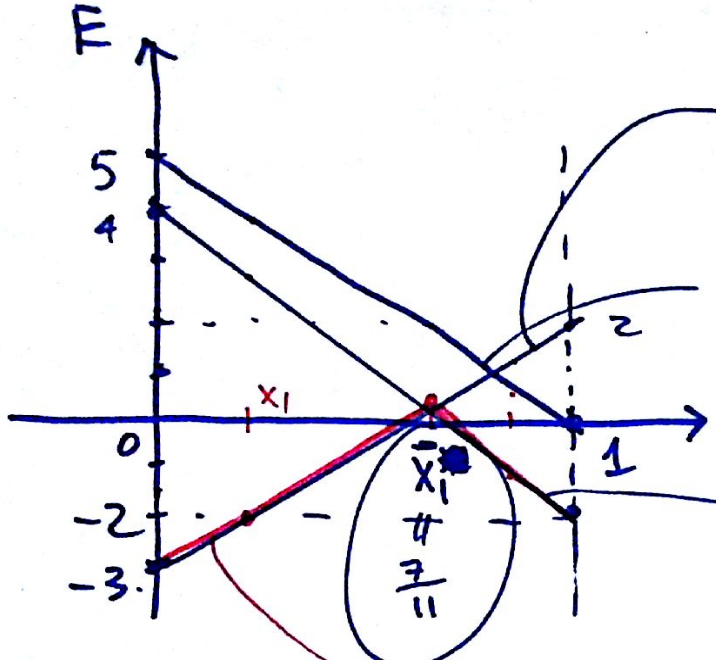
$$F(x) = \min_y E(x,y) = \min_y \left(y_1 E_{x_1}(x_1) + y_2 E_{x_2}(x_1) + y_3 E_{x_3}(x_1) \right)$$

$$\overset{\text{by Lemma}}{=} \min_{1\leq j\leq 3} \{ E_{x_j}(x_1) \}$$

Graph each pure strategy payoff: $E = E_{x_1}(x_1) = 5(1-x_1)$, $0 \leq x_1 \leq 1$

$E = E_{x_2}(x_1) = 4 - 6x_1$,

$E = E_{x_3}(x_1) = -3 + 5x_1$,



$E = E_{x_3}(x_1), \quad y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
$= -3 + 5x_1$

$E = E_{x_1}(x_1) = 5(1 - x_1), \quad y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$E = E_{x_2}(x_1) = 4 - 6x_1, \quad y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Sketch $F(x) = \min\limits_{y} E(x,y) = \min\limits_{1 \le j \le 3} \{ E_{x_j}(x_1) \}$.

$E = F(x_1) = \min\limits_{y} E(x,y)$.

$$E(\bar{x}, \bar{y}) = \max_{x} \left(\min_{y} E(x,y) \right)$$

$$= \max_{0 \le x_1 \le 1} \left(F(x_1) \right) = F(x_1^*) \overset{\text{or}}{=} F(\bar{x}_1).$$

$\bar{x}_1$ is the intersection of $\begin{cases} E = E_{x_3}(x_1) = -3 + 5x_1 \\ E = E_{x_2}(x_1) = 4 - 6x_1 \end{cases}$.

$\Rightarrow -3 + 5x_1 = 4 - 6x_1 \Rightarrow \bar{x}_1 = \frac{7}{11}$.