$$Ax + Ix_s = b \implies \begin{aligned} a_1 x &+ x_{n+1} &&= b_1 \\ a_2 x &+ \quad x_{n+2} &&= b_2 \\ &\vdots \\ a_m x &+ \qquad x_{n+m} &&= b_m. \end{aligned}$$

By row operations, at any step of the simplex method, including the last step, the 0th equation must be of the form.

$$z - c^T x + \underline{y_1 a_1 x + y_1 x_{n+1}} + \underline{y_2 a_2 x + y_2 x_{n+2}} + \cdots + y_m a_m x + y_m x_{m+m}$$
$$= y_1 b_1 + y_2 b_2 + \cdots + y_m b_m.$$

for which y_i is the scalar used to multiply row i and then add to row 0.

$$\Rightarrow z - c^T x + [y_1, \cdots, y_m]\begin{bmatrix} a_1 x \\ a_2 x \\ \vdots \\ a_m x \end{bmatrix} + [y_1, \cdots, y_m]\overbrace{\begin{bmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}}^{x_s} = [y_1, \cdots, y_m]\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$z - c^T x + y^T \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} x + y^T x_s = y^T b$$

$$\Rightarrow z - c^T x + y^T A x + y^T x_s = y^T b.$$

$$\Rightarrow z = (c^T - y^T A)x - y^T x_s + y^T b$$

when the optimization test is satisfied, the coefficients of x, x_s must be ≤ 0, with the optimal value to be $z^* = y^T b$. I.e

$$c^T - y^T A \leq 0, \quad -y^T \leq 0$$

i.e. $y^T A \geq c^T, \quad y^T \geq 0. \implies A^T y \geq c, \ y \geq 0$

<u>Thm</u>: The primal standard LP. max $z = c^T x$, sub. to $Ax \le b$, $x \ge 0$ has solution iff there is a $y \ge 0$, s.t. $A^T y \ge c$ and $w = y^T b = b^T y$ is minimal, i.e. the dual standard LP problem: min: $w = b^T y$, sub. to: $A^T y \ge c$, $y \ge 0$.

"pf" $\Rightarrow$. We have, $\exists\, y \ge 0$, $A^T y \ge c$. Need to show is that $w = b^T y$ is minimal. Otherwise, $\exists\, \bar{y} \ge 0$, $A^T \bar{y} \ge c$, s.t. $b^T \bar{y} < b^T y$. $\Rightarrow$ lead to contraction. ... $\square$.

$\Leftarrow$ similar.

Duality Theory: ~~$\exists$ solution to max~~

$$\begin{aligned} &\max: z = c^T x && && \min: w = b^T y \\ &\text{sub. } Ax \le b && \Longleftrightarrow && \text{sub. } A^T y \ge c \\ &\qquad x \ge 0 && && \qquad y \ge 0. \end{aligned}$$

and y of the dual is the shadow price of the primal.

$$z = (c^T - y^T A)x - \boxed{y^T} x_s + y^T b. \text{ i.e } z^* = y^T b.$$

i.e. it is the coefficients of the slack variables in absolute values.

$$z^* = y_1 b_1 + y_2 b_2 + \cdots + y_m b_m.$$

The rate of return: $\dfrac{\partial z^*}{\partial b_i} = y_i$.

Also known as Lagrange multipliers for y_i.

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§14. Game Theory.

Ex: P.R.S. paper. rock, scissor.

Two players game. player 1, P1. strategies. $s_1, s_2, \cdots s_n$

player 2. P_2. $t_1, t_2, \cdots t_m$.

Payoff for P1 (class)

P2 (me)

	s_1	s_2	s_3	$\cdots$	s_n
t_1	\$2	-\$1			
t_2	a_{21}		a_{ij}		
$\vdots$					
t_m	a_{m1}				

Payoff for P2

P_2

	s_1	s_2	$\cdots$	s_n ← st. for P1.
t_1	b_{11}			
t_2	b_{21}		b_{ij}	
$\vdots$				
t_m				

<u>Zero Sum Game</u>: let $A = [a_{ij}]_{m\times n}$ be the payoff ~~for~~ matrix for player 1. Then ~~for~~ the payoff matrix for player 2 is exactly the opposite: $-A$.

Ex,

Payoff P1

	P	R	S
P	0	−1	+1
R	+1	0	−1
S	−1	+	0

Payoff matrix for P1 : $A_1 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$.

For zero sum game,

in $\$$. Payoff matrix for P2 : $A_2 = -A_1$

$$= \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$