Simplex method

(1)

- Standard LP : max: $z = c_1x_1 + \cdots + c_nx_n = c^T x$, $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

  sub. to: $Ax \le b$, $A = [a_{ij}]_{m \times n}$, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

  $x \ge 0$

- Augmented LP: max: $z = c^T x$

  sub. to: $Ax + x_s = b$ with slack variable $x_s = \begin{bmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}$

  $x \ge 0,\ x_s \ge 0$

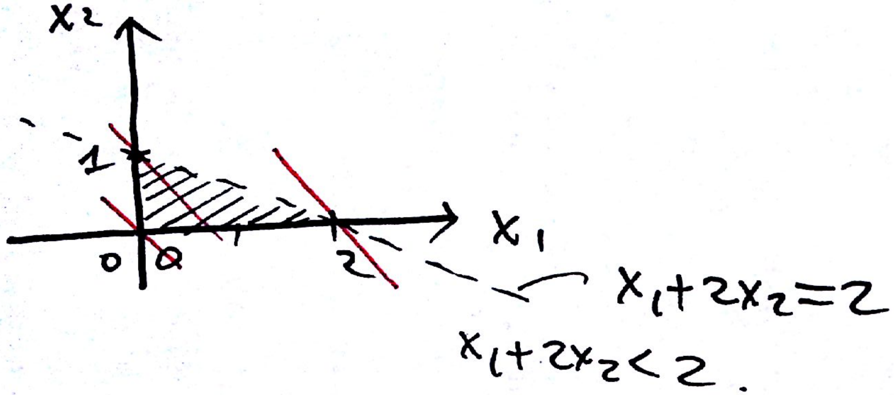
Thm: Standard LP is equivalent to Aug. LP.

"pf" For Aug. LP. Feasible region $= \{ Ax + x_s = b,\ x \ge 0,\ x_s \ge 0 \}$. from m-constraints in n+m space, which is an n-dimensional ~~subspa~~ subset in $\mathbb{R}^{n+m}$. I.e. $[A \vdots I]$ is the ~~cor~~ coeff. matrix of the Aug. LP. By the rank theorem, $[A \vdots I]\begin{bmatrix} x \\ x_s \end{bmatrix} = b$ is of $n = (n+m) - \text{rank}([A \vdots I])$ dimensional. It is a n-d. polygone, for which the linear optimization problem to max $z = c^T x$ takes place on the boundary, and corner of the polygone.

Ex:

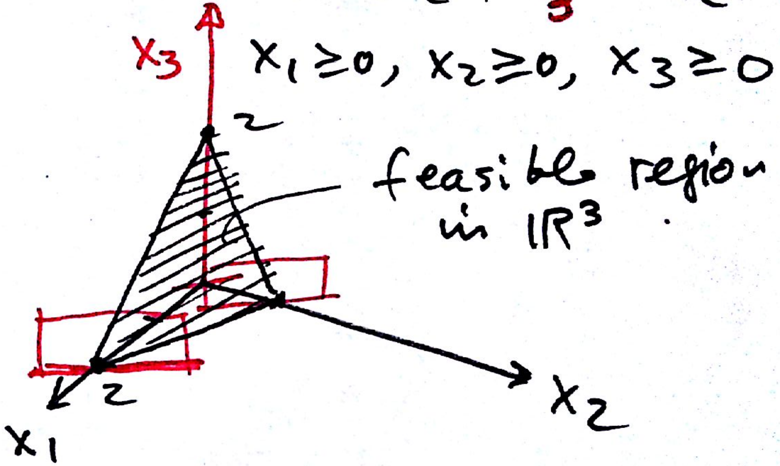
S. LP.

$$\max \ z = x_1 + x_2$$
$$\text{sub. to} \quad x_1 + 2x_2 \leq 2$$
$$x_1 \geq 0,\ x_2 \geq 0 .$$



Aug. LP

$$\max \ z = x_1 + x_2$$
$$\text{sub. to} \quad x_1 + 2x_2 + x_3 = 2$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 .$$



Rmk:

$$z - c^T x = z_0 \quad \text{w/ max. } z_0 \iff \text{Aug. L.P.}$$
$$[A \vdots I]\begin{bmatrix} x \\ x_s \end{bmatrix} = b$$
$$x \geq 0,\ x_s \geq 0 .$$

- solution to Aug. LP $\iff$ solution to its row equivalent form.

Def: Let $\left\{ \begin{array}{l} z - c^T\bar{x} = z_0 \\ \cancel{[A \mid I]\begin{bmatrix} x \\ x_s \end{bmatrix} = b} \\ \cancel{x \geq 0,\ x_s \geq 0} \end{array} \right.$ with $\bar{x} = \begin{bmatrix} x \\ x_s \end{bmatrix}$. be any row-equivalent form

$A\bar{x} = b$
$\bar{x} = \begin{bmatrix} x \\ x_s \end{bmatrix} \geq 0$

to ~~Aug~~. LP. Let $R = \begin{array}{ccc} z & \bar{x} & \\ \left[\begin{array}{c} 1, \\ 0 \end{array}\right. & \begin{array}{c} -\tilde{c}^T, \\ A \end{array} & \left.\begin{array}{c} z_0 \\ b \end{array}\right] \end{array}$ be the

coefficient matrix. R is a feasible echelon form if

→ (1) $b \geq 0$

(2) $\text{rank}(A) = m$ where $A = [a_{ij}]_{m \times n}$ matrix

i.e $m = \dim(\text{row}(A))$.

(3) Each row of A has one entry, $a_{ij_i} = 1$, and for the corresponding j_ith column of R, all other column entries are zeroes, i.e $a_{kj_i} = 0$, $k \neq i$.

Rmk: $R = \begin{array}{c} \begin{array}{cccccc} z & x_1 & x_2 & x_3 & x_4 & \text{rhs} \end{array} \\ \left[\begin{array}{c|ccccc} \boxed{1} & 0 & -1 & 0 & 3 & -5 \\ 0 & 0 & 3 & \boxed{1} & -1 & 3 \\ 0 & \boxed{1} & 0 & 0 & 0 & 2 \end{array}\right] \end{array}$ — feasible echelon form.

~~Find~~ Make up your own nonfeasible eqn. matrix.

(k)

- Let R be the feasible echelon form. Then the solution $\bar{x} = \begin{bmatrix} x \\ x_S \end{bmatrix}$ with $x_{j_i} = b_i \geq 0$ and $x_j = 0$ for $j \neq j_i$, $i = 1, 2, \cdots, m$. is a feasible solution. In fact, each $x_j = 0$, $j \neq j_i$, it represents one boundry of the feasible region, and w/ m many such $x_j = 0$, we have a corner point of the f. region. And the correspontig z-value is z_0.

- By definition, the correspondig $(x_{1j_1}, x_{2j_2}, \cdots, x_{mj_m})$ is called the ~~bai~~ basic variable, and x_j w/ $j \neq j_i$, $i = 1, 2, \cdots, m$, are the non-basic variables. This designation is relative to a particular feasible echelon form.

<u>Simplex method</u> is to go from one basic feasible point (solution) (i.e one feasible echelon form) to an adjacent basic feasible point (i.e another feasible echelon form) untill the optimality condition (i.e the optimal solution) is reached

$$z = \bar{c}^T \bar{x} + \bar{z}_0$$

with $\bar{c} \le 0$, i.e. for the coefficients of basic variables $x_{j_i} = b_i \ge 0$, $\bar{c}_{j_i} = 0$ and for the nonbasic variable x_j, $j \ne j_i$, $c_j \le 0$

Ex: $z = -x_1 - x_2 + \bar{z}_0 \Rightarrow$ opt. soln. $x_1 = x_2 = 0$.